Contents

Chapter 1

VECTOR CALCULUS

The physical quantities encountered in fluid mechanics can be classified into three classes: (a) *scalars*, such as pressure, density, viscosity, temperature, length, mass, volume and time; (b) *vectors*, such as velocity, acceleration, displacement, linear momentum and force, and (c) *tensors*, such as stress, rate of strain and vorticity tensors.

Scalars are completely described by their *magnitude* or absolute value, and they do not require direction in space for their specification. In most cases, we shall denote scalars by lower case lightface italic type, such as p for pressure and ρ for density. Operations with scalars, i.e., addition and multiplication, follow the rules of elementary algebra. A *scalar field* is a real-valued function that associates a scalar (i.e., a real number) with each point of a given region in space. Let us consider, for example, the *right-handed Cartesian coordinate system* of Fig. 1.1 and a closed three-dimensional region V occupied by a certain amount of a moving fluid at a given time instance t. The density ρ of the fluid at any point (x, y, z) of V defines a scalar field denoted by $\rho(x, y, z)$. If the density is, in addition, time-dependent, one may write $\rho{=}\rho(x, y, z, t)$.

Vectors are specified by their *magnitude* and their *direction* with respect to a given frame of reference. They are often denoted by lower case boldface type, such as $\mathbf{u}$ for the velocity vector. A *vector field* is a vector-valued function that associates a vector with each point of a given region in space. For example, the velocity of the fluid in the region V of Fig. 1.1 defines a vector field denoted by $\mathbf{u}(x, y, z, t)$. A vector field which is independent of time is called a *steady-state* or *stationary* vector field. The magnitude of a vector $\mathbf{u}$ is designated by $|\mathbf{u}|$ or simply by u.

Vectors can be represented geometrically as arrows; the direction of the arrow specifies the direction of the vector and the length of the arrow, compared to some chosen scale, describes its magnitude. Vectors having the same length and the same direction, regardless of the position of their initial points, are said to be *equal*. A vector having the same length but the opposite direction to that of the vector $\mathbf{u}$ is denoted by $-\mathbf{u}$ and is called the *negative* of $\mathbf{u}$.

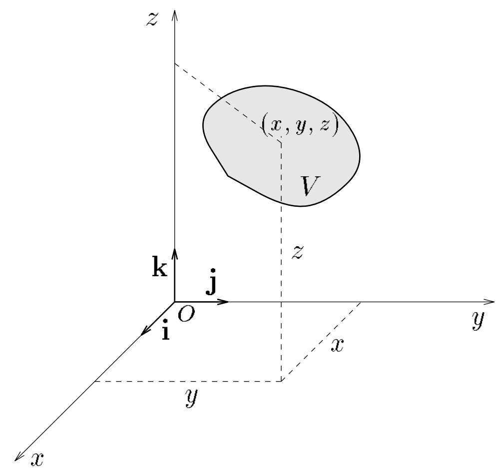

Figure 1.1. *Cartesian system of coordinates.*

The *sum* (or the *resultant*) **u**+**v** of two vectors **u** and **v** can be found using the *parallelogram law* for vector addition, as shown in Fig. 1.2a. Extensions to sums of more than two vectors are immediate. The difference **u**-**v** is defined as the sum **u**+(−**v**); its geometrical construction is shown in Fig. 1.2b.

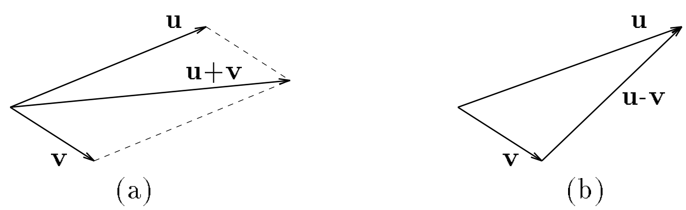

Figure 1.2. *Addition and subtraction of vectors.*

The vector of length zero is called the *zero vector* and is denoted by **0**. Obviously, there is no natural direction for the zero vector. However, depending on the problem, a direction can be assigned for convenience. For any vector **u**,

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

and

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0} \,.$$

Vector addition obeys the *commutative* and *associative* laws. If **u**, **v** and **w** are vectors, then

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative law
$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative law

If $\mathbf{u}$ is a nonzero vector and m is a nonzero scalar, then the *product* $m\mathbf{u}$ is defined as the vector whose length is $|m|$ times the length of $\mathbf{u}$ and whose direction is the same as that of $\mathbf{u}$ if $m > 0$, and opposite to that of $\mathbf{u}$ if $m < 0$. If m=0 or $\mathbf{u}$=$\mathbf{0}$, then $m\mathbf{u}$=$\mathbf{0}$. If $\mathbf{u}$ and $\mathbf{v}$ are vectors and m and n are scalars, then

$m\mathbf{u} = \mathbf{u}m$	Commutative law
$m(n\mathbf{u}) = (mn)\mathbf{u}$	Associative law
$(m + n)\mathbf{u} = m\mathbf{u} + n\mathbf{u}$	Distributive law
$m(\mathbf{u} + \mathbf{v}) = m\mathbf{u} + m\mathbf{v}$	Distributive law

Note also that $(-1)\mathbf{u}$ is just the negative of $\mathbf{u}$,

$$(-1)\mathbf{u} = -\mathbf{u}\,.$$

A *unit vector* is a vector having unit magnitude. The three vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ which have the directions of the positive x, y and z axes, respectively, in the Cartesian coordinate system of Fig. 1.1 are unit vectors.

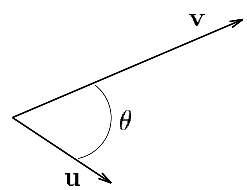

Figure 1.3. *Angle between vectors* $\mathbf{u}$ *and* $\mathbf{v}$*.*

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in a two- or three-dimensional space positioned so that their initial points coincide (Fig. 1.3). The *angle* θ between $\mathbf{u}$ and $\mathbf{v}$ is the angle determined by $\mathbf{u}$ and $\mathbf{v}$ that satisfies $0 \leq \theta \leq \pi$. The *dot product* (or *scalar product*) of $\mathbf{u}$ and $\mathbf{v}$ is a scalar quantity defined by

$$\mathbf{u} \cdot \mathbf{v} \equiv uv \cos\theta\,. \tag{1.1}$$

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors and m is a scalar, then

$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	Commutative law
$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$	Distributive law
$m(\mathbf{u} \cdot \mathbf{v}) = (m\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (m\mathbf{v})$	

Moreover, the dot product of a vector with itself is a positive number that is equal to the square of the length of the vector:

$$\mathbf{u}\cdot\mathbf{u} \;=\; u^2 \quad\Longleftrightarrow\quad u \;=\; \sqrt{\mathbf{u}\cdot\mathbf{u}}\,. \tag{1.2}$$

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors and

$$\mathbf{u}\cdot\mathbf{v} \;=\; 0\,,$$

then $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* or *perpendicular* to each other.

A vector set $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is said to be an *orthogonal set* or *orthogonal system* if every distinct pair of the set is orthogonal, i.e.,

$$\mathbf{u}_i\cdot\mathbf{u}_j \;=\; 0\,, \quad i \neq j\,.$$

If, in addition, all its members are unit vectors, then the set $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is said to be *orthonormal.* In such a case,

$$\mathbf{u}_i\cdot\mathbf{u}_j \;=\; \delta_{ij}\,, \tag{1.3}$$

where δ_{ij} is the *Kronecker delta*, defined as

$$\delta_{ij} \;\equiv\; \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{1.4}$$

The three unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ defining the Cartesian coordinate system of Fig. 1.1 form an orthonormal set:

$$\boxed{\begin{array}{c} \mathbf{i}\cdot\mathbf{i} \;=\; \mathbf{j}\cdot\mathbf{j} \;=\; \mathbf{k}\cdot\mathbf{k} \;=\; 1 \\[2ex] \mathbf{i}\cdot\mathbf{j} \;=\; \mathbf{j}\cdot\mathbf{k} \;=\; \mathbf{k}\cdot\mathbf{i} \;=\; 0 \end{array}} \tag{1.5}$$

The *cross product* (or *vector product* or *outer product*) of two vectors $\mathbf{u}$ and $\mathbf{v}$ is a vector defined as

$$\mathbf{u}\times\mathbf{v} \;\equiv\; uv\sin\theta\,\mathbf{n}\,, \tag{1.6}$$

where $\mathbf{n}$ is the unit vector normal to the plane of $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{n}$ form a *right-handed* orthogonal system, as illustrated in Fig. 1.4. The magnitude of $\mathbf{u}\times\mathbf{v}$ is the same as that of the area of a parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\sin\theta$=0 and $\mathbf{u}\times\mathbf{v}=\mathbf{0}$. For instance, $\mathbf{u}\times\mathbf{u}=\mathbf{0}$.

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors and m is a scalar, then

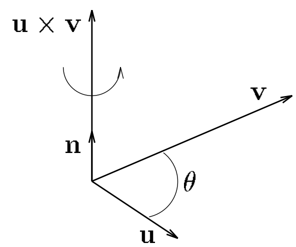

Figure 1.4. *The cross product* $\mathbf{u}\times\mathbf{v}$.

$\mathbf{u}\times\mathbf{v} \;=\; -\,\mathbf{v}\times\mathbf{u}$	Not commutative
$\mathbf{u}\times(\mathbf{v}+\mathbf{w}) \;=\; \mathbf{u}\times\mathbf{v} \;+\; \mathbf{u}\times\mathbf{w}$	Distributive law
$m(\mathbf{u}\times\mathbf{v}) \;=\; (m\mathbf{u})\times\mathbf{v} \;=\; \mathbf{u}\times(m\mathbf{v}) \;=\; (\mathbf{u}\times\mathbf{v})m$	

For the three unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ one gets:

$$\begin{array}{c} \mathbf{i}\times\mathbf{i} \;=\; \mathbf{j}\times\mathbf{j} \;=\; \mathbf{k}\times\mathbf{k} \;=\; \mathbf{0}\,, \\ \mathbf{i}\times\mathbf{j} \;=\; \mathbf{k}\,, \quad \mathbf{j}\times\mathbf{k} \;=\; \mathbf{i}\,, \quad \mathbf{k}\times\mathbf{i} \;=\; \mathbf{j}\,, \\ \mathbf{j}\times\mathbf{i} \;=\; -\mathbf{k}\,, \quad \mathbf{k}\times\mathbf{j} \;=\; -\mathbf{i}\,, \quad \mathbf{i}\times\mathbf{k} \;=\; -\mathbf{j}\,. \end{array} \tag{1.7}$$

Note that the cyclic order $(\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{i},\mathbf{j},\cdots)$, in which the cross product of any neighboring pair in order is the next vector, is consistent with the right-handed orientation of the axes as shown in Fig. 1.1.

The product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is called the *scalar triple product* of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, and is a scalar representing the volume of a parallelepiped with $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ as the edges. The product $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ is a vector called the *vector triple product.* The following laws are valid:

$(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w} \;\neq\; \mathbf{u}\,(\mathbf{v}\cdot\mathbf{w})$	Not associative
$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \;\neq\; (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$	Not associative
$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \;=\; (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} \;-\; (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}$	
$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} \;=\; (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} \;-\; (\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}$	
$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) \;=\; \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) \;=\; \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$	

Thus far, we have presented vectors and vector operations from a geometrical viewpoint. These are treated analytically in Section 1.2.

1.1 Systems of Coordinates

A coordinate system in the three-dimensional space is defined by choosing a set of three *linearly independent* vectors, $B=\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, representing the three fundamental directions of the space. The set B is a *basis* of the three-dimensional space, i.e., each vector $\mathbf{v}$ of this space is uniquely written as a *linear combination* of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$:

$$\mathbf{v} \;=\; v_1\,\mathbf{e}_1 \;+\; v_2\,\mathbf{e}_2 \;+\; v_3\,\mathbf{e}_3 \,. \tag{1.8}$$

The scalars v_1, v_2 and v_3 are the *components* of $\mathbf{v}$ and represent the magnitudes of the *projections* of $\mathbf{v}$ onto each of the fundamental directions. The vector $\mathbf{v}$ is often denoted by $\mathbf{v}(v_1, v_2, v_3)$ or simply by (v_1, v_2, v_3).

In most cases, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are *unit* vectors. In the three coordinate systems that are of interest in this book, i.e., *Cartesian*, *cylindrical* and *spherical* coordinates, the three vectors are, in addition, orthogonal. Hence, in all these systems, the basis $B=\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is orthonormal:

$$\mathbf{e}_i \cdot \mathbf{e}_j \;=\; \delta_{ij} \,. \tag{1.9}$$

(In some cases, nonorthogonal systems are used for convenience; see, for example, [1].) For the cross products of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, one gets:

$$\mathbf{e}_i \times \mathbf{e}_j \;=\; \sum_{k=1}^{3} \epsilon_{ijk}\,\mathbf{e}_k \,, \tag{1.10}$$

where ϵ_{ijk} is the *permutation symbol*, defined as

$$\epsilon_{ijk} \;\equiv\; \begin{cases} 1\,, & \text{if } ijk\text{=123, 231, or 312 (i.e, an even permutation of 123)} \\ -1\,, & \text{if } ijk\text{=321, 132, or 213 (i.e, an odd permutation of 123)} \\ 0\,, & \text{if any two indices are equal} \end{cases} \tag{1.11}$$

A useful relation involving the permutation symbol is the following:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \;=\; \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\,a_i b_j c_k \,. \tag{1.12}$$

The Cartesian (or *rectangular*) system of coordinates (x, y, z), with

$$-\infty < x < \infty \,, \quad -\infty < y < \infty \quad \text{and} \quad -\infty < z < \infty \,,$$

has already been introduced, in previous examples. Its basis is often denoted by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. The decomposition of a vector $\mathbf{v}$ into its three components

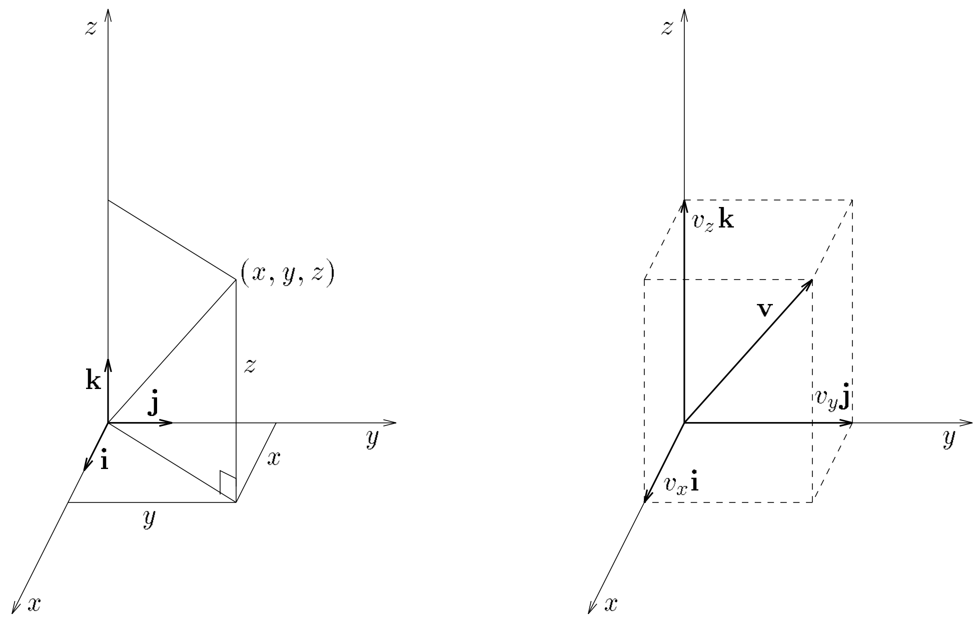

Figure 1.5. *Cartesian coordinates* (x, y, z) *with* $-\infty < x < \infty$, $-\infty < y < \infty$ *and* $-\infty < z < \infty$.

$(r, \theta, z) \longrightarrow (x, y, z)$	$(x, y, z) \longrightarrow (r, \theta, z)$
Coordinates	
$x = r\cos\theta$	$r = \sqrt{x^2 + y^2}$
$y = r\sin\theta$	$\theta = \begin{cases} \arctan\frac{y}{x}, & x > 0,\ y \geq 0 \\ \pi + \arctan\frac{y}{x}, & x < 0 \\ 2\pi + \arctan\frac{y}{x}, & x > 0,\ y < 0 \end{cases}$
$z = z$	$z = z$
Unit vectors	
$\mathbf{i} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta$	$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$
$\mathbf{j} = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta$	$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$
$\mathbf{k} = \mathbf{e}_z$	$\mathbf{e}_z = \mathbf{k}$

Table 1.1. *Relations between Cartesian and cylindrical polar coordinates.*

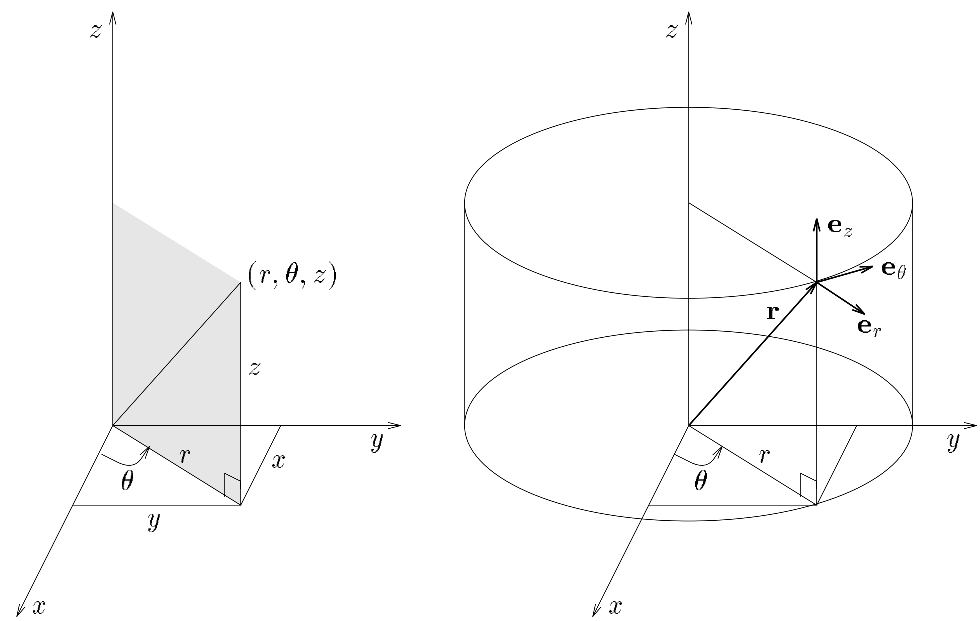

Figure 1.6. *Cylindrical polar coordinates* (r, θ, z) *with* $r \geq 0$, $0 \leq \theta < 2\pi$ *and* $-\infty < z < \infty$, *and the position vector* $\mathbf{r}$.

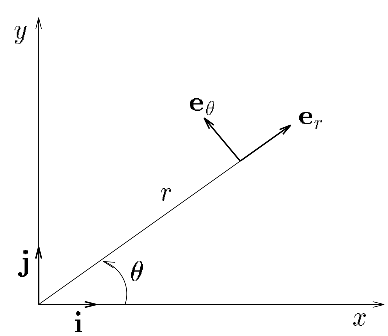

Figure 1.7. *Plane polar coordinates* (r, θ).

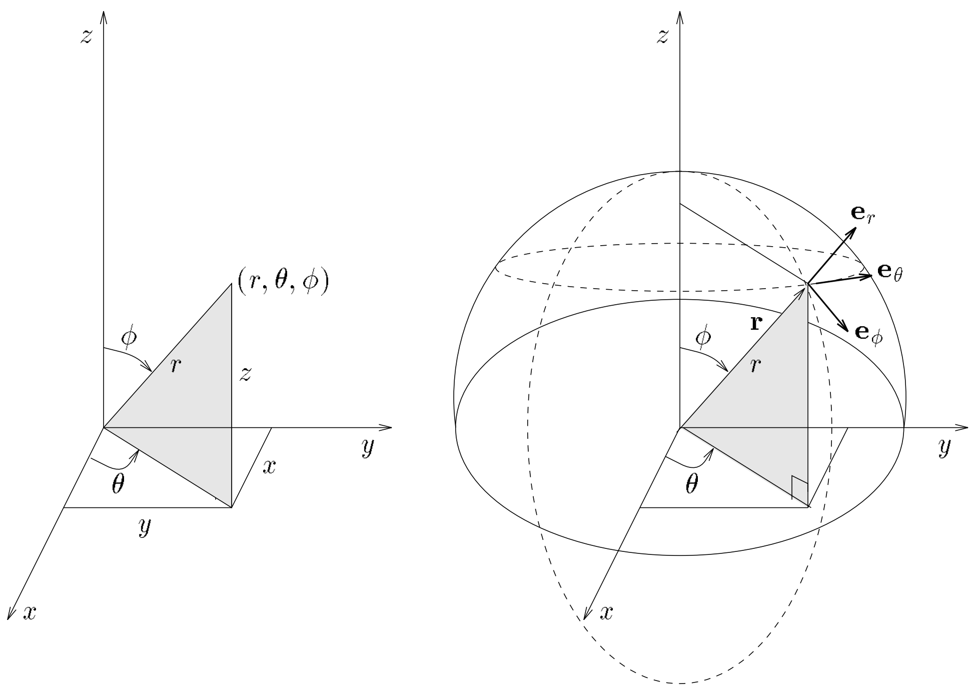

Figure 1.8. *Spherical polar coordinates* (r, θ, ϕ) *with* $r \geq 0$, $0 \leq \phi \leq \pi$ *and* $0 \leq \theta \leq 2\pi$, *and the position vector* $\mathbf{r}$.

$(r,\theta,\phi) \longrightarrow (x,y,z)$	$(x,y,z) \longrightarrow (r,\theta,\phi)$
Coordinates	
$x = r\sin\phi\cos\theta$	$r = \sqrt{x^2+y^2+z^2}$
$y = r\sin\phi\sin\theta$	$\phi = \begin{cases} \arctan\frac{\sqrt{x^2+y^2}}{z}, & z>0 \\ \frac{\pi}{2}, & z=0 \\ \pi + \arctan\frac{\sqrt{x^2+y^2}}{z}, & z<0 \end{cases}$
$z = r\cos\phi$	$\theta = \begin{cases} \arctan\frac{y}{x}, & x>0,\ y\geq 0 \\ \pi+\arctan\frac{y}{x}, & x<0 \\ 2\pi+\arctan\frac{y}{x}, & x>0,\ y<0 \end{cases}$
Unit vectors	
$\mathbf{i} = \sin\phi\cos\theta\,\mathbf{e}_r + \cos\phi\cos\theta\,\mathbf{e}_\phi - \sin\theta\,\mathbf{e}_\theta$	$\mathbf{e}_r = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$
$\mathbf{j} = \sin\phi\sin\theta\,\mathbf{e}_r + \cos\phi\sin\theta\,\mathbf{e}_\phi + \cos\theta\,\mathbf{e}_\theta$	$\mathbf{e}_\phi = \cos\phi\cos\theta\,\mathbf{i} + \cos\phi\sin\theta\,\mathbf{j} - \sin\phi\,\mathbf{k}$
$\mathbf{k} = \cos\phi\,\mathbf{e}_r - \sin\phi\,\mathbf{e}_\phi$	$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$

Table 1.2. *Relations between Cartesian and spherical polar coordinates.*

(v_x, v_y, v_z) is depicted in Fig. 1.5. It should be noted that, throughout this book, we use *right-handed* coordinate systems.

The cylindrical and spherical polar coordinates are the two most important orthogonal *curvilinear* coordinate systems. The cylindrical polar coordinates (r, θ, z), with

$$r \geq 0\,, \quad 0 \leq \theta < 2\pi \quad \text{and} \quad -\infty < z < \infty\,,$$

are shown in Fig. 1.6 together with the Cartesian coordinates sharing the same origin. The basis of the cylindrical coordinate system consists of three orthonormal vectors: the radial vector $\mathbf{e}_r$, the azimuthal vector $\mathbf{e}_\theta$, and the axial vector $\mathbf{e}_z$. Note that the azimuthal angle θ revolves around the z axis. Any vector $\mathbf{v}$ is decomposed into, and is fully defined by its components $\mathbf{v}(v_r, v_\theta, v_z)$ with respect to the cylindrical system. By invoking simple trigonometric relations, any vector, including those of the bases, can be transformed from one system to another. Table 1.1 lists the formulas for making coordinate conversions from cylindrical to Cartesian coordinates and vice versa.

On the xy plane, i.e., if the z coordinate is ignored, the cylindrical polar coordinates are reduced to the familiar *plane polar coordinates* (r, θ) shown in Fig. 1.7.

The spherical polar coordinates (r, θ, ϕ), with

$$r \geq 0\,, \quad 0 \leq \phi \leq \pi \quad \text{and} \quad 0 \leq \theta < 2\pi\,,$$

together with the Cartesian coordinates with the same origin, are shown in Fig. 1.8. It should be emphasized that r in cylindrical and spherical coordinates is not the same. The basis of the spherical coordinate system consists of three orthonormal vectors: the radial vector $\mathbf{e}_r$, the meridional vector $\mathbf{e}_\phi$, and the azimuthal vector $\mathbf{e}_\theta$. Any vector $\mathbf{v}$ can be decomposed into the three components, $\mathbf{v}(v_r, v_\theta, v_\phi)$, which are the scalar projections of $\mathbf{v}$ onto the three fundamental directions. The transformation of a vector from spherical to Cartesian coordinates (sharing the same origin) and vice-versa obeys the relations of Table 1.2.

The choice of the appropriate coordinate system, when studying a fluid mechanics problem, depends on the geometry and symmetry of the flow. Flow between parallel plates is conveniently described by Cartesian coordinates. *Axisymmetric* (i.e., *axially symmetric*) flows, such as flow in an annulus, are naturally described using cylindrical coordinates, and flow around a sphere is expressed in spherical coordinates. In some cases, nonorthogonal systems might be employed too. More details on other coordinate systems and transformations can be found elsewhere [1].

Example 1.1.1. Basis of the cylindrical system

Show that the basis B=$\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ of the cylindrical system is orthonormal.

Solution:

Since $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k}$=1 and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i}$=0, we obtain:

$$\begin{aligned}
\mathbf{e}_r\cdot\mathbf{e}_r &= (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\cdot(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \;=\; \cos^2\theta + \sin^2\theta = 1\\
\mathbf{e}_\theta\cdot\mathbf{e}_\theta &= (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\cdot(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) \;=\; \sin^2\theta + \cos^2\theta = 1\\
\mathbf{e}_z\cdot\mathbf{e}_z &= \mathbf{k}\cdot\mathbf{k} = 1\\
\mathbf{e}_r\cdot\mathbf{e}_\theta &= (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\cdot(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) \;=\; 0\\
\mathbf{e}_r\cdot\mathbf{e}_z &= (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\cdot\mathbf{k} \;=\; 0\\
\mathbf{e}_\theta\cdot\mathbf{e}_z &= (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\cdot\mathbf{k} \;=\; 0
\end{aligned}$$

□

Example 1.1.2. The position vector

The *position vector* $\mathbf{r}$ defines the position of a point in space, with respect to a coordinate system. In Cartesian coordinates,

$$\mathbf{r} \;=\; x\,\mathbf{i} \,+\, y\,\mathbf{j} \,+\, z\,\mathbf{k}\,, \tag{1.13}$$

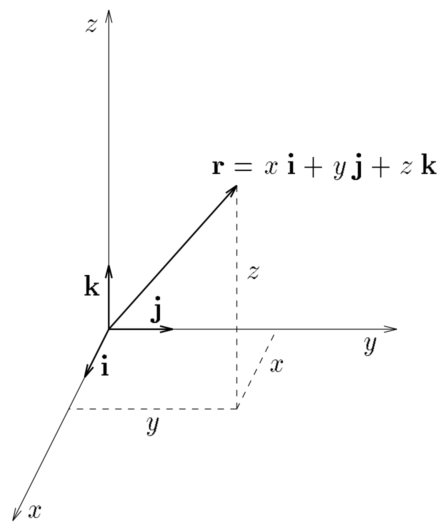

Figure 1.9. *The position vector,* **r**, *in Cartesian coordinates.*

and thus

$$|\mathbf{r}| \;=\; (\mathbf{r}\cdot\mathbf{r})^{\frac{1}{2}} \;=\; \sqrt{x^2+y^2+z^2}\,. \tag{1.14}$$

The decomposition of **r** into its three components (x, y, z) is illustrated in Fig. 1.9.

In cylindrical coordinates, the position vector is given by

$$\mathbf{r} \;=\; r\,\mathbf{e}_r \;+\; z\,\mathbf{e}_z \quad \text{with} \quad |\mathbf{r}| \;=\; \sqrt{r^2+z^2}\,. \tag{1.15}$$

Note that the magnitude $|\mathbf{r}|$ of the position vector *is not* the same as the radial cylindrical coordinate r. Finally, in spherical coordinates,

$$\mathbf{r} \;=\; r\,\mathbf{e}_r \quad \text{with} \quad |\mathbf{r}| \;=\; r\,, \tag{1.16}$$

that is, $|\mathbf{r}|$ is the radial spherical coordinate r. Even though expressions (1.15) and (1.16) for the position vector are obvious (see Figs. 1.6 and 1.8, respectively), we will derive both of them, starting from Eq. (1.13) and using coordinate transformations.

In cylindrical coordinates,

$$\begin{aligned}
\mathbf{r} \;&=\; x\,\mathbf{i} \;+\; y\,\mathbf{j} \;+\; z\,\mathbf{k} \\
&=\; r\cos\theta\,(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) \;+\; r\sin\theta\,(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) \;+\; z\,\mathbf{e}_z \\
&=\; r\,(\cos^2\theta + \sin^2\theta)\,\mathbf{e}_r \;+\; r\,(-\sin\theta\cos\theta + \sin\theta\cos\theta)\,\mathbf{e}_\theta \;+\; z\,\mathbf{e}_z \\
&=\; r\,\mathbf{e}_r \;+\; z\,\mathbf{e}_z\,.
\end{aligned}$$

In spherical coordinates,

$$\begin{aligned}
\mathbf{r} &= x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \\
&= r\sin\phi\cos\theta\,(\sin\phi\cos\theta\,\mathbf{e}_r + \cos\phi\cos\theta\,\mathbf{e}_\phi - \sin\theta\,\mathbf{e}_\theta) \\
&\quad + r\sin\phi\sin\theta\,(\sin\phi\sin\theta\,\mathbf{e}_r + \cos\phi\sin\theta\,\mathbf{e}_\phi + \cos\theta\,\mathbf{e}_\theta) \\
&\quad + r\cos\phi\,(\cos\phi\,\mathbf{e}_r - \sin\phi\,\mathbf{e}_\phi) \\
&= r\,[\sin^2\phi\,(\cos^2\theta + \sin^2\theta) \;+\; \cos^2\phi]\,\mathbf{e}_r \\
&\quad + r\sin\phi\cos\phi\,[(\cos^2\theta + \sin^2\theta) - 1]\,\mathbf{e}_\phi \\
&\quad + r\sin\phi\,(-\sin\theta\cos\theta + \sin\theta\cos\theta)\,\mathbf{e}_\theta \\
&= r\,\mathbf{e}_r\,.
\end{aligned}$$

□

Example 1.1.3. Derivatives of the basis vectors

The basis vectors **i**, **j** and **k** of the Cartesian coordinates are fixed and do not change with position. This is not true for the basis vectors in curvilinear coordinate systems. From Table 1.1, we observe that, in cylindrical coordinates,

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \quad \text{and} \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\,;$$

therefore, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ change with θ. Taking the derivatives with respect to θ, we obtain:

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} = \mathbf{e}_\theta$$

and

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j} = -\mathbf{e}_r\,.$$

All the other spatial derivatives of $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are zero. Hence,

$$\boxed{\begin{array}{lll}
\frac{\partial \mathbf{e}_r}{\partial r} = \mathbf{0} & \frac{\partial \mathbf{e}_\theta}{\partial r} = \mathbf{0} & \frac{\partial \mathbf{e}_z}{\partial r} = \mathbf{0} \\
\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta & \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r & \frac{\partial \mathbf{e}_z}{\partial \theta} = \mathbf{0} \\
\frac{\partial \mathbf{e}_r}{\partial z} = \mathbf{0} & \frac{\partial \mathbf{e}_\theta}{\partial z} = \mathbf{0} & \frac{\partial \mathbf{e}_z}{\partial z} = \mathbf{0}
\end{array}} \tag{1.17}$$

Similarly, for the spatial derivatives of the unit vectors in spherical coordinates, we obtain:

$$
\begin{array}{lll}
\frac{\partial \mathbf{e}_r}{\partial r} = \mathbf{0} & \frac{\partial \mathbf{e}_\phi}{\partial r} = \mathbf{0} & \frac{\partial \mathbf{e}_\theta}{\partial r} = \mathbf{0} \\
\frac{\partial \mathbf{e}_r}{\partial \phi} = \mathbf{e}_\phi & \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_r & \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \mathbf{0} \\
\frac{\partial \mathbf{e}_r}{\partial \theta} = \sin\phi\, \mathbf{e}_\theta & \frac{\partial \mathbf{e}_\phi}{\partial \theta} = \cos\phi\, \mathbf{e}_\theta & \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\sin\phi\, \mathbf{e}_r - \cos\phi\, \mathbf{e}_\phi
\end{array}
\quad (1.18)
$$

Equations (1.17) and (1.18) are very useful in converting differential operators from Cartesian to orthogonal curvilinear coordinates.

□

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