PERGAMON Computers and Mathematics with Applications 41 (2001) 647–657

An International Journal
computers & mathematics with applications

www.elsevier.nl/locate/camwa

Some Aspects of the One-Dimensional Version of the Method of Fundamental Solutions

Y.-S. SMYRLIS, A. KARAGEORGHIS AND G. GEORGIOU
Department of Mathematics and Statistics, University of Cyprus
P.O. Box 20537, 1678 Nicosia, Cyprus

(Received February 2000; accepted July 2000)

Abstract—The method of fundamental solutions (MFS) is a well-established boundary-type numerical method for the solution of certain two- and three-dimensional elliptic boundary value problems [1,2]. The basic ideas were introduced by Kupradze and Alexidze (see, e.g., [3]), whereas the present form of the MFS was proposed by Mathon and Johnston [4]. The aim of this work is to investigate the one-dimensional analogue of the MFS for the solution of certain two-point boundary value problems. In particular, the one-dimensional MFS is formulated in the case of linear scalar ordinary differential equations of even degree with constant coefficients. A mathematical justification for the method is provided and various aspects related to its applicability from both an analytical and a numerical standpoint are examined. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Method of fundamental solutions, Two-point boundary value problems.

1. INTRODUCTION

We consider the linear two-point boundary value problem

$$\mathbf{y}' = A(x)\mathbf{y}, \qquad x \in (\alpha, \beta), \tag{1.1}$$

where $\mathbf{y} : (\alpha, \beta) \to \mathbb{R}^n$, $A(x)$ is an $n \times n$ matrix, subject to the boundary conditions

$$B_\alpha \mathbf{y}(\alpha) + B_\beta \mathbf{y}(\beta) = \mathbf{c}, \tag{1.2}$$

where B_α and B_β are constant matrices and $\mathbf{y}^\top = [y_1, y_2, \dots, y_n]$. If the matrix $Y(x)$ is a fundamental matrix of the system of differential equations, that is, $Y' = AY$ and Y nonsingular, then the solution of the problem is

$$\mathbf{y}(x) = Y(x)\,\mathbf{d} = Y(x)Q^{-1}\mathbf{c}, \tag{1.3}$$

where the matrix Q is given from

$$Q = B_\alpha Y(\alpha) + B_\beta Y(\beta). \tag{1.4}$$

PII: S0898-1221(00)00308-4

Provided the matrix Q is nonsingular, there exists a unique solution to the two-point boundary value problem.

Let us now consider the scalar n^{th}-order two-point boundary value problem

$$\begin{gathered} d_n u^{(n)} + \cdots + d_0 u^{(0)} = 0, \qquad x \in (\alpha, \beta), \\ Bu = \mathbf{c}, \end{gathered} \tag{1.5}$$

where $Bu = \mathbf{c}$ corresponds to the n-boundary conditions at the points $x = \alpha$ and $x = \beta$,

$$\sum_{j=1,\dots,n} b^{\alpha}_{i,j} u^{(j-1)}(\alpha) + \sum_{j=1,\dots,n} b^{\beta}_{i,j} u^{(j-1)}(\beta) = c_i, \qquad i = 1, \dots, n. \tag{1.6}$$

If we set

$$y_i = u^{(i-1)}, \qquad i = 1, \dots, n, \tag{1.7}$$

and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -\dfrac{d_0}{d_n} & -\dfrac{d_1}{d_n} & -\dfrac{d_2}{d_n} & \cdots & -\dfrac{d_{n-1}}{d_n} \end{pmatrix}, \tag{1.8}$$

then (1.5) becomes equivalent to (1.1),(1.2). (See [5,6].)

Clearly, there could be numerical complications if the matrix Q given by (1.4) is poorly conditioned. We also observe that if $Y(x)$ is a fundamental matrix and P a nonsingular constant matrix, then $Y(x)P$ is also a fundamental matrix.

2. THE ONE-DIMENSIONAL MFS

The solution of the scalar equation is expressible in terms of fundamental kernels (which in the one-dimensional case play the role of fundamental solutions as in [4,7,8])

$$\varphi(x) = \sum_{j=1}^{n} \sigma^p_j K_j(x, x_p) + \sum_{j=1}^{n} \sigma^q_j K_j(x, x_q), \tag{2.1}$$

where x_p and x_q are points outside the interval $[\alpha, \beta]$ with

$$x_p < \alpha < \beta < x_q \tag{2.2}$$

and the family of *fundamental kernels* $\{K_j(x, y)\}_{j=1,\dots,n}$ spans the space of solutions of $Lu = 0$, i.e., the set $\{K_j(x, x_p), K_j(x, x_q)\}_{j=1,\dots,n}$ is a fundamental set of solutions. The kernels are expected to be of the form

$$K_j(x, y) = \kappa_j(|x - y|).$$

Such a set of fundamental kernels exists for differential operators of even degree with constant coefficients which is the case under consideration. (See Proposition 3.) From now on, we shall assume that the order of the linear scalar operator is $2n$.

The fundamental matrix of the equivalent first-order linear system is $Y(x) = (Y_{i,j})_{i,j=1,\dots,n}$ and consists of the elements

$$\begin{aligned} Y_{i,j} &= K^{(i-1)}_j(x, x_p), \qquad j = 1, \dots, n, \quad i = 1, \dots, 2n \quad \text{and} \\ Y_{i,n+j} &= K^{(i-1)}_j(x, x_q), \qquad j = 1, \dots, n, \quad i = 1, \dots, 2n. \end{aligned}$$

The solution in terms of fundamental solutions is given by

$$\mathbf{y}(x) = Y(x)\boldsymbol{\sigma} = Y(x)Q^{-1}\mathbf{c}, \tag{2.3}$$

where now $\boldsymbol{\sigma}^\top = [\sigma_1^p, \dots, \sigma_n^p, \sigma_1^q, \dots, \sigma_n^q]$. A case of particular interest is when the matrices B_α, B_β are diagonal with

$$(B_\alpha)_{ii} = \begin{cases} 1, & \text{if } i = 1, \dots, n, \\ 0, & \text{if } i = n+1, \dots, 2n, \end{cases}$$
$$(B_\beta)_{ii} = \begin{cases} 0, & \text{if } i = 1, \dots, n, \\ 1, & \text{if } i = n+1, \dots, 2n. \end{cases}$$

In this case, the solution and its first $n-1$ derivatives are prescribed at the end points $x = \alpha$ and $x = \beta$.

3. SPECIAL CASES

We first consider two-point boundary value problems in which the governing equation is

$$Lu = \frac{d^{2n}u}{dx^{2n}} = 0. \tag{3.1}$$

In particular, we are concerned with expressing the solution of the boundary value problem

$$\begin{gathered} u^{(2n)} = 0, \\ u^{(j)}(\alpha) = a_j, \qquad u^{(j)}(\beta) = b_j, \quad j = 0, \dots, n-1 \end{gathered} \tag{3.2}$$

in terms of fundamental solutions for different values of x_p and x_q.

The above boundary value problem is always nonsingular.

PROPOSITION 1. *The boundary value problem (3.2) has a unique solution for any n and any choice of the constants a_j and b_j, $j = 1, \dots, n$.*

PROOF. It is sufficient to show that (3.2) has a solution for every choice of a_j, b_j where all but one are zero and the nonzero element is equal to one.

We observe that $\varphi(x) = (x-\alpha)^{n-1}(x-\beta)^n$ satisfies the differential equation and the boundary conditions

$$\varphi^{(k)}(\alpha) = 0, \quad k = 0, \dots, n-2 \qquad \text{and} \qquad \varphi^{(k)}(\beta) = 0, \quad k = 0, \dots, n-1,$$

but

$$\varphi^{(n-1)}(\alpha) \neq 0.$$

By setting

$$\varphi_{\alpha,n-1}(x) = \frac{\varphi(x)}{\varphi^{(n-1)}(\alpha)},$$

then $\varphi_{\alpha,n-1}(x)$ and all its derivatives up to order $n-2$ vanish at $x = \alpha$ and up to order $n-1$ at $x = \beta$. The $(n-1)^{\text{th}}$ derivative at $x = \alpha$ is equal to one.

The function $\varphi_{\alpha,n-2}(x)$ satisfying $\varphi_{\alpha,n-2}^{(2n)} = 0$ and the boundary conditions

$$\varphi_{\alpha,n-2}^{(j)}(\alpha) = \delta_{j,n-2} \qquad \text{and} \qquad \varphi_{\alpha,n-2}^{(j)}(\beta) = 0, \qquad j = 0, \dots, n-1,$$

could then be constructed as a linear combination of $(x-a)^{n-2}(x-b)^n$ and $\varphi_{a,n-1}(x)$.

In this manner, we can construct the set of functions $\mathcal{B} = \{\varphi_{\alpha,j}, \varphi_{\beta,j}\}_{j=0,\dots,n-1}$ which form a basis for the space of solutions of $u^{(2n)} = 0$. In particular, the solution of (3.2) could be written as

$$\varphi(x) = \sum_{j=0,\dots,n-1} a_j \varphi_{\alpha,j}(x) + \sum_{j=0,\dots,n-1} b_j \varphi_{\beta,j}(x).$$

Uniqueness follows from the observation that if

$$\psi(x) = \sum_{j=0,\dots,n-1} c_{\alpha,j} \varphi_{\alpha,j}(x) + \sum_{j=0,\dots,n-1} c_{\beta,j} \varphi_{\beta,j}(x),$$

then

$$\psi^{(j)}(\alpha) = c_{\alpha,j}, \quad \psi^{(j)}(\beta) = c_{\beta,j}, \qquad j = 0, \dots, n-1,$$

which completes the proof. ∎

A natural[1] set of fundamental kernels for (3.2) is

$$K_j(x,y) = \frac{1}{2} \frac{|x-y|^{2j-1}}{(2j-1)!}, \qquad j = 1, \dots, n. \tag{3.3}$$

PROPOSITION 2. *The set of kernels $\{K_1, \dots, K_n\}$ constitutes a fundamental set of kernels for the equation $u^{(2n)} = 0$ for every $x_p \neq x_q$.*

PROOF. It is sufficient to show that the monomials $1, x, \dots, x^{2n-1}$, which constitute a fundamental set of solutions of the equation $u^{(2n)} = 0$, are spanned by the set of kernels $\{K_1, \dots, K_n\}$. More specifically, we should show that they are spanned by the functions

$$K_j(x, x_p) = \frac{1}{2} \frac{(x-x_p)^{2j-1}}{(2j-1)!}, \quad K_j(x, x_q) = \frac{1}{2} \frac{(x_q - x)^{2j-1}}{(2j-1)!}, \qquad j = 1, \dots, n.$$

This can be proved inductively. For $j = 1$, the functions

$$K_1(x, x_p) = \frac{1}{2}(x - x_p), \qquad K_1(x, x_q) = \frac{1}{2}(x_q - x),$$

span the space of all polynomials of degree less than 2. Assuming that the functions $\{K_j(x,x_p), K_j(x,x_q)\}_{j=1,\dots,l}$ span the space of all polynomials of degree less than $2l$, then clearly the degree of $p(x) = K_{l+1}(x,x_p) - K_{l+1}(x,x_q)$ is $2l$, whereas the degree of $K_j(x,x_p)$ is $2l+1$. Therefore, the polynomials $\{K_j(x,x_p), K_j(x,x_q)\}_{j=1,\dots,l+1}$ span the space of all polynomials of degree less than $2l+2$ and the proof is completed. ∎

3.1. Example 1

Let us consider the simplest possible case of

$$\begin{aligned} u'' &= 0, \qquad & x \in (\alpha, \beta), \\ u(\alpha) &= g_0, \qquad & u(\beta) = g_1. \end{aligned} \tag{3.4}$$

[1]The coefficients of the kernels are chosen in order to satisfy the following. If u is given by the formula

$$u(x) = \int_\alpha^\beta K_j(x,y) f(y)\, dy = (\mathcal{K}_j f)(x),$$

where f is continuous, then u satisfies

$$u^{(2j)}(x) = f(x),$$

i.e., $LK_j = \delta(x-y)$ and the integral operator $\mathcal{K}_j$ is a right inverse of the differential operator $\frac{d^{2j}}{dx^{2j}}$.

The analytical solution of this problem is

$$u(x) = \frac{g_1 - g_0}{\beta - \alpha} x + \frac{\beta g_1 - \alpha g_0}{\beta - \alpha}. \tag{3.5}$$

The fundamental solution of the operator $L \equiv \frac{d^2}{dx^2}$ is

$$K_1(x, x_p) = \frac{1}{2} |x - x_p|, \tag{3.6}$$

and the solution can be expressed in terms of fundamental solutions as [4]

$$u(x) = K_1(x, x_p)\sigma_1^p + K_1(x, x_q)\sigma_1^q, \qquad x \in [\alpha, \beta]. \tag{3.7}$$

The imposition of the boundary conditions yields

$$\begin{aligned} K_1(\alpha, x_p)\sigma_1^p + K_1(\alpha, x_q)\sigma_1^q &= g_0, \\ K_1(\beta, x_p)\sigma_1^p + K_1(\beta, x_q)\sigma_1^q &= g_1. \end{aligned}$$

The solution of the system gives

$$\sigma_1^p = \frac{g_0 K_1(\beta, x_q) - g_1 K_1(\alpha, x_q)}{K_1(\alpha, x_p) K_1(\beta, x_q) - K_1(\alpha, x_q) K_1(\beta, x_p)} \tag{3.8}$$

and

$$\sigma_1^q = \frac{g_1 K_1(\alpha, x_p) - g_0 K_1(\beta, x_p)}{K_1(\alpha, x_p) K_1(\beta, x_q) - K_1(\alpha, x_q) K_1(\beta, x_p)}. \tag{3.9}$$

Substitution of these expressions in the MFS solution leads to the identical expression we have for the exact solution. There is cancellation of all terms involving x_q and x_p.

Consider now the mixed boundary value problem

$$\begin{aligned} u'' &= 0, & x &\in (\alpha, \beta), \\ u(\alpha) &= g_0, & u'(\beta) &= g_1', \end{aligned} \tag{3.10}$$

the analytical solution of which is

$$u(x) = g_1' x + g_0 - \alpha g_1'. \tag{3.11}$$

The analysis of the MFS solution is similar to the Dirichlet case. The satisfaction of the boundary conditions leads to the following expressions:

$$\sigma_1^p = \frac{g_0 K_1'(\beta, x_q) - g_1' K_1(\alpha, x_q)}{K_1(\alpha, x_p) K_1'(\beta, x_q) - K_1(\alpha, x_q) K_1'(\beta, x_p)} \tag{3.12}$$

and

$$\sigma_1^q = \frac{g_1' K_1(\alpha, x_p) - g_0 K_1'(\beta, x_p)}{K_1(\alpha, x_p) K_1'(\beta, x_q) - K_1(\alpha, x_q) K_1'(\beta, x_p)}. \tag{3.13}$$

Substitution of σ_1^p and σ_1^q in the MFS solution leads again to the exact solution.

3.2. Example 2

Consider now the fourth-order equation

$$u'''' = 0, \qquad x \in (\alpha, \beta), \tag{3.14}$$

subject to the boundary conditions

$$u(\alpha) = g_0, \qquad u'(\alpha) = g_0', \qquad u(\beta) = g_1, \qquad u'(\beta) = g_1'. \tag{3.15}$$

The analytical solution of the above problem is

$$u(x) = g_0 + g_0'(x-\alpha) + \left(\frac{3(g_1-g_0)}{\beta-\alpha} - 2g_0' - g_1'\right)\frac{(x-\alpha)^2}{\beta-\alpha} + \left(g_0' + g_1' - 2\frac{g_1-g_0}{\beta-\alpha}\right)\frac{(x-\alpha)^3}{(\beta-\alpha)^2}.$$

The fundamental solution of the operator $L \equiv \frac{d^4}{dx^4}$ is

$$K_2(x, x_p) = \frac{1}{12}|x - x_p|^3. \tag{3.16}$$

The solution can be expressed in terms of the fundamental solutions of the biharmonic and Laplace operators as [9], i.e.,

$$u(x) = K_2(x, x_p)\,\sigma_2^p + K_2(x, x_q)\sigma_2^q + K_1(x, x_p)\,\sigma_1^p + K_1(x, x_q)\sigma_1^q, \qquad x \in [\alpha, \beta]. \tag{3.17}$$

The imposition of the boundary conditions leads to a 4×4 system of the form

$$A\mathbf{x} = \mathbf{c}, \tag{3.18}$$

where

$$A = \begin{pmatrix} \frac{1}{12}(\alpha - x_p)^3 & \frac{1}{12}(x_q - \alpha)^3 & \frac{1}{2}(\alpha - x_p) & \frac{1}{2}(x_q - \alpha) \\ \frac{1}{4}(\alpha - x_p)^2 & -\frac{1}{4}(x_q - \alpha)^2 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{12}(\beta - x_p)^3 & \frac{1}{12}(x_q - \beta)^3 & \frac{1}{2}(\beta - x_p) & \frac{1}{2}(x_q - \beta) \\ \frac{1}{4}(\beta - x_p)^2 & -\frac{1}{4}(x_q - \beta)^2 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

$\mathbf{x} = [\sigma_2^p, \sigma_2^q, \sigma_1^p, \sigma_1^q]^\top$, and $\mathbf{c} = [g_0, g_0', g_1, g_1']^\top$.

4. MORE GENERAL EXAMPLES

In this section, consider two-point boundary value problems in which the governing equation is of the form

$$Lu = \sum_{j=0}^{n} w_j u^{(2j)} = 0,$$

where w_j are constants and $w_n \neq 0$. It can be shown that such operators always possess a set of fundamental kernels.

PROPOSITION 3. *Let $L = \sum_{j=0}^{n} w_j \frac{d^{2j}}{dx^{2j}}$ be a differential operator and $p_L(t) = \sum_{j=0}^{n} w_j t^{2j}$ its characteristic polynomial with roots as follows:*

1. *real: $\pm r_j$, $j = 1, \dots, \rho$ with multiplicities μ_j, $j = 1, \dots, \rho$, respectively,*
2. *purely imaginary (nonzero): $\pm i s_j$, $j = \rho+1, \dots, \rho+\sigma$ with multiplicities μ_j, $j = \rho + 1, \dots, \rho+\sigma$, respectively,*
3. *complex (nonreal and not imaginary): $\pm r_j \pm i s_j$, $j = \rho+\sigma+1, \dots, \rho+\sigma+\tau$ with multiplicities μ_j, $j = \rho+\sigma+1, \dots, \rho+\sigma+\tau$, respectively,*

where $\rho + \sigma + 2\tau = n$. Then the kernels

$$K_{j,\nu}(x,y) = \frac{1}{2}\frac{|x-y|^{\nu-1}}{(\nu-1)!}\frac{e^{r_j|x-y|}}{r_j}, \tag{4.1}$$

with $j = 1, \dots, \rho$, $\nu = 1, \dots, \mu_j$,

$$K_{j,\nu}(x,y) = \frac{1}{2}\frac{|x-y|^{\nu-1}}{(\nu-1)!}\frac{\sin(s_j|x-y|)}{s_j}, \tag{4.2}$$

with $j = \rho+1, \dots, \rho+\sigma$, $\nu = 1, \dots, \mu_j$, and

$$\begin{aligned} K^c_{j,\nu}(x,y) &= \frac{1}{2}\frac{|x-y|^{\nu-1}}{(\nu-1)!}\frac{e^{r_j|x-y|}\cos(s_j|x-y|)}{\sqrt{r_j^2+s_j^2}}, \\ K^s_{j,\nu}(x,y) &= \frac{1}{2}\frac{|x-y|^{\nu-1}}{(\nu-1)!}\frac{e^{r_j|x-y|}\sin(s_j|x-y|)}{\sqrt{r_j^2+s_j^2}}, \end{aligned} \tag{4.3}$$

with $j = \rho+\sigma+1, \dots, \rho+\sigma+\tau$, $\nu = 1, \dots, \mu_j$, constitute a fundamental set of kernels for L at the points x_p and x_q provided

$$x_q - x_p \neq \sum_{j=\rho+1}^{\rho+\sigma+\tau} \frac{\kappa_j \pi}{s_j}, \qquad \text{with } k_j \in \mathbb{Z}. \tag{4.4}$$

PROOF. It is sufficient to show that the above kernels span the space of solutions of $Lu = 0$. More specifically, it is sufficient to show that the functions

$$\varphi_{j,\nu,p}(x) = K_{j,\nu}(x,x_p), \qquad \varphi_{j,\nu,q}(x) = K_{j,\nu}(x,x_q),$$

with $j = 1, \dots, \rho+\sigma$ and $\nu = 1, \dots, \mu_j$, together with the functions

$$\begin{aligned} \psi_{j,\nu,c,p}(x) &= K^c_{j,\nu}(x,x_p), & \psi_{j,\nu,c,q}(x) &= K^c_{j,\nu}(x,x_q), \\ \psi_{j,\nu,s,p}(x) &= K^s_{j,\nu}(x,x_p), & \psi_{j,\nu,s,q}(x) &= K^s_{j,\nu}(x,x_q), \end{aligned}$$

constitute a fundamental set of solutions of $Lu = 0$. This can be readily derived from Proposition 2. ∎

4.1. Example 1

We consider the two-point boundary value problem

$$\begin{aligned} u'' + \lambda^2 u = 0, &\qquad x \in (\alpha, \beta), \\ u(\alpha) = g_0, &\qquad u(\beta) = g_1, \end{aligned} \tag{4.5}$$

the analytical solution of which is

$$u(x) = \frac{(g_0\cos\lambda\beta - g_1\cos\lambda\alpha)\sin\lambda x + (g_1\sin\lambda\alpha - g_0\sin\lambda\beta)\cos\lambda x}{\sin\lambda(\alpha-\beta)}. \tag{4.6}$$

The fundamental solution of the operator $L \equiv \frac{d^2}{dx^2} + \lambda^2$ is [8]

$$K_{1,1}(x,x_p) = \frac{1}{2\lambda}\sin\lambda|x-x_p|. \tag{4.7}$$

The imposition of the boundary conditions leads to following expressions for σ_1^p and σ_1^q:

$$\sigma_1^p = \frac{g_0 \sin\lambda(x_q-\beta) - g_1 \sin\lambda(x_q-\alpha)}{\sin\lambda\,(\alpha-x_p)\sin\lambda(x_q-\beta) - \sin\lambda(x_q-\alpha)\sin\lambda\,(\beta-x_p)}, \tag{4.8}$$

$$\sigma_1^q = \frac{g_1 \sin\lambda\,(\alpha-x_p) - g_0 \sin\lambda\,(\beta-x_p)}{\sin\lambda\,(\alpha-x_p)\sin\lambda(x_q-\beta) - \sin\lambda(x_q-\alpha)\sin\lambda\,(\beta-x_p)}. \tag{4.9}$$

Problems in the solution occur for the values of x_p and x_q for which the determinant of the system vanishes, i.e., when

$$D = \sin\lambda\,(\alpha-x_p)\sin\lambda(x_q-\beta) - \sin\lambda(x_q-\alpha)\sin\lambda\,(\beta-x_p) = 0$$

or

$$\cos\lambda\,(x_q-x_p+\beta-\alpha) - \cos\lambda\,(x_q-x_p-\beta+\alpha) = 0.$$

This occurs when

$$x_q - x_p = \frac{n\pi}{\lambda}, \qquad n \in \mathbb{N}. \tag{4.10}$$

In the case of the mixed boundary value problem

$$\begin{aligned} u'' + \lambda^2 u &= 0, \qquad & x &\in (\alpha,\beta), \\ u(\alpha) &= g_0, \qquad & u'(\beta) &= g_1', \end{aligned} \tag{4.11}$$

the analysis of the MFS solution is similar to the Dirichlet case. The potentially troublesome determinant,

$$D = \lambda\sin\lambda(\alpha-x_q)\cos\lambda\,(\beta-x_p) - \lambda\sin\lambda\,(\alpha-x_p)\cos\lambda(\beta-x_q),$$

this vanishes when

$$\sin\lambda\,(x_p-x_q-\beta+\alpha) - \sin\lambda\,(x_q-x_p-\beta+\alpha) = 0,$$

or when

$$x_q - x_p = \frac{n\pi}{\lambda}, \qquad n \in \mathbb{N}. \tag{4.12}$$

Finally, in the case of the Neumann problem

$$u'' + \lambda^2 u = 0, \qquad x \in (\alpha,\beta), \tag{4.13}$$

$$u'(\alpha) = g_0', \qquad u'(\beta) = g_1', \tag{4.14}$$

the determinant of the system,

$$D = \lambda^2\cos\lambda(\alpha-x_q)\cos\lambda\,(\beta-x_p) - \lambda^2\cos\lambda\,(\alpha-x_p)\cos\lambda(\beta-x_q)$$

vanishes when

$$\cos\lambda\,(\alpha-\beta+x_p-x_q) - \cos\lambda\,(\alpha-\beta-x_p+x_q) = 0,$$

i.e., when

$$x_q - x_p = \frac{n\pi}{\lambda}, \qquad n \in \mathbb{N}. \tag{4.15}$$

These restrictions on the choice of x_p and x_q appear because of the fact that the functions

$$\varphi_1(x) = K_{1,1}\,(x,x_p) \qquad \text{and} \qquad \varphi_2(x) = K_{1,1}(x,x_q)$$

are linearly dependent if

$$\lambda\,(x_q-x_p) = \kappa\pi, \qquad \kappa \in \mathbb{N}.$$

4.2. Example 2

We consider the homogeneous rod problem [10]

$$u'''' - \lambda^4 u = 0, \qquad x \in (\alpha, \beta), \tag{4.16}$$

subject to the boundary conditions

$$u(\alpha) = g_0, \qquad u'(\alpha) = g_0', \qquad u(\beta) = g_1, \qquad u'(\beta) = g_1'. \tag{4.17}$$

The analytical solution is of the form

$$u(x) = R \sin \lambda x + S \cos \lambda x + T e^{\lambda x} + V e^{-\lambda x},$$

where the constants R, S, T, and V can be determined from the boundary conditions. The solution can be expressed in terms of the fundamental solutions

$$u(x) = K_{1,1}(x, x_p)\,\sigma_1^p + K_{1,1}(x, x_q)\sigma_1^q + K_{2,1}(x, x_p)\,\sigma_2^p + K_{2,1}(x, x_q)\sigma_2^q, \qquad x \in [\alpha, \beta],$$

where

$$K_{1,1}(x, y) = \frac{e^{-\lambda|x-y|}}{\lambda}$$

and

$$K_{2,1}(x, y) = \frac{\sin(\lambda|x-y|)}{\lambda}.$$

The satisfaction of the boundary conditions leads to a 4×4 system $A\mathbf{x} = \mathbf{c}$, where $\mathbf{x} = [\sigma_1^p, \sigma_1^q, \sigma_2^p, \sigma_2^q]^\top$,

$$A = \begin{pmatrix} \frac{1}{\lambda} e^{-\lambda(\alpha - x_p)} & \frac{1}{\lambda} e^{-\lambda(x_q - \alpha)} & \frac{1}{\lambda} \sin \lambda\,(\alpha - x_p) & \frac{1}{\lambda} \sin \lambda(x_q - \alpha) \\ -e^{-\lambda(\alpha - x_p)} & e^{-\lambda(x_q - \alpha)} & \cos \lambda\,(\alpha - x_p) & -\cos \lambda(x_q - \alpha) \\ \frac{1}{\lambda} e^{-\lambda(\beta - x_p)} & \frac{1}{\lambda} e^{-\lambda(x_q - \beta)} & \frac{1}{\lambda} \sin \lambda\,(\beta - x_p) & \frac{1}{\lambda} \sin \lambda(x_q - \beta) \\ -e^{-\lambda(\beta - x_p)} & e^{-\lambda(x_q - \beta)} & \cos \lambda\,(\beta - x_p) & -\cos \lambda(x_q - \beta) \end{pmatrix}$$

and $\mathbf{c} = [g_0, g_0', g_1, g_1']^\top$. Problems in the MFS solution occur when the determinant of the system $D = \det(A)$ vanishes. It can be found that this occurs when

$$x_q - x_p = \frac{n\pi}{\lambda}, \qquad n \in \mathbb{N}. \tag{4.18}$$

As in the previous example, these restrictions on the choice of x_p and x_q appear because of the fact that

$$\varphi_1(x) = K_{2,1}(x, x_p) \qquad \text{and} \qquad \varphi_2(x) = K_{2,1}(x, x_q)$$

are linearly dependent if

$$\lambda\,(x_q - x_p) = \kappa\pi, \qquad \kappa \in \mathbb{N}.$$

4.3. Example 3

A two-point boundary value problem which is even more pathologically ill-posed is the one governed by the equation

$$u^{(4)} + \left(\lambda^2 + \mu^2\right) u^{(2)} + \lambda^2\mu^2 u = 0. \tag{4.19}$$

In this case, the fundamental kernels

$$K_{1,1}(x, y) = \frac{\sin(\lambda|x-y|)}{2\lambda} \qquad \text{and} \qquad K_{2,1}(x, y) = \frac{\sin(\mu|x-y|)}{2\mu}$$

span the solutions of (4.19) only when

$$x_q - x_p \neq \frac{\kappa_1\pi}{\lambda} + \frac{\kappa_2\pi}{\mu}, \qquad \text{for } \kappa_1, \kappa_2 \in \mathbb{N}.$$

Remarkably, in the case $\lambda/\mu \notin \mathbb{Q}$, the set of pairs (x_p, x_q) for which our problem is ill-posed are dense in $\mathbb{R}^2$.

5. NUMERICAL RESULTS

In order to examine the influence of the position of the sources x_p and x_q on the conditioning of the matrix resulting from the application of the MFS and the accuracy of the MFS solution, we considered the two-point boundary value problem (3.2) in the specific case when $\alpha = -1$, $\beta = 1$, for $n = 1, 2, 3, 4$. In each case, the boundary conditions were taken to correspond to the exact solution $u(x) = x^{2n-1}$. The sources x_p and x_q were placed symmetrically at $x_p = -1 - \epsilon$ and $x_q = 1 + \epsilon$ where ϵ was taken to be a variable parameter. The calculated MFS solution was compared with the analytical solution and the error calculated at 101 equidistant points on $[-1, 1]$. For each value of ϵ, we calculated the largest absolute error in the solution. We also examined the condition number of the matrix A as ϵ was varied. In particular, we calculated an estimate for the condition number κ_A of A in the L^∞ norm using the NAG pair F07ADF-AGF [11]. The graph of the condition number estimate versus ϵ for $n = 1, 2, 3, 4$ is presented in Figure 1. It was observed that the condition number estimate behaves like $O(\epsilon^{4n-3})$. In Figure 2, we plot the maximum absolute error versus ϵ for $n = 1, 2, 3, 4$. In this case, we observed that the maximum absolute error behaves like $O(\epsilon^{2n-1})$. Similar results can be observed in the case of more general examples studied in Section 4.

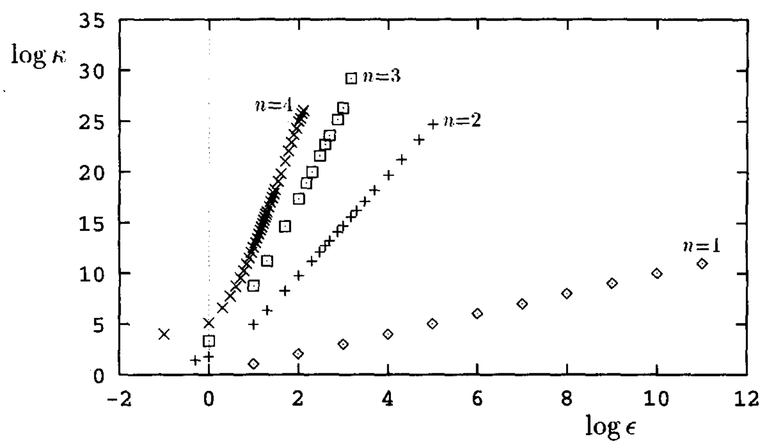

Figure 1. Log-log plot of the condition number κ vs. ϵ.

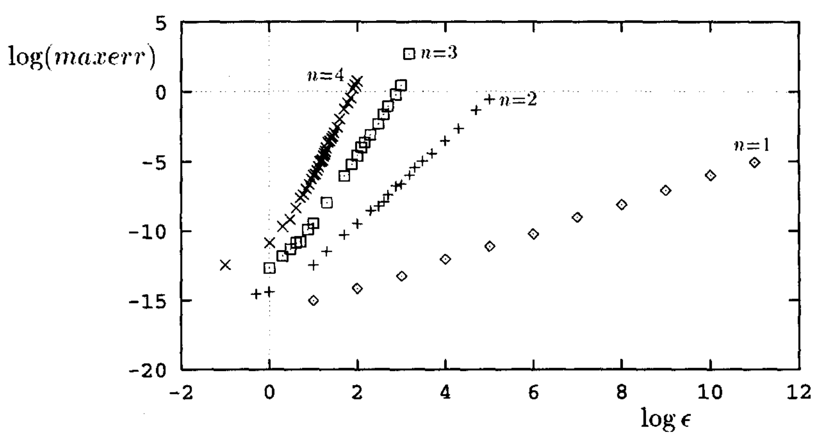

Figure 2. Log-log plot of the maximum error vs. ϵ.

REFERENCES

1. G. Fairweather and A. Karageorghis, The method of fundamental solutions for elliptic boundary value problems, *Adv. Comput. Math.* **9**, 69–95, (1998).
2. M.A. Golberg and C.S. Chen, The method of fundamental solutions for potential, Helmholtz and diffusion problems, In *Boundary Integral Methods: Numerical and Mathematical Aspects*, Chapter 4, (Edited by M.A. Golberg), pp. 105–176, WIT Press and Computational Mechanics Publications, Boston, MA, (1999).

3. V.D. Kupradze, *Potential Methods in the Theory of Elasticity*, Israel Program for Scientific Translations, Jerusalem, (1965).
4. R. Mathon and R.L. Johnston, The approximate solution of elliptic boundary-value problems by fundamental solutions, *SIAM J. Numer. Anal.* **14**, 638–650, (1977).
5. U.M. Ascher, R.M.M. Mattheij and R.D. Russell, *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, SIAM, Philadelphia, PA, (1995).
6. E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, (1955).
7. P.K. Banerjee and R. Butterfield, *Boundary Element Methods in Engineering Science*, McGraw-Hill, Maidenhead, (1981).
8. P.K. Kythe, *Fundamental Solutions for Differential Operators and Applications*, Birkhäuser, Boston, MA, (1996).
9. M. Maiti and S.K. Chakrabarti, Integral equations solutions for simply supported polygonal plates, *Internat. J. Eng. Sci.* **12**, 793–806, (1974).
10. R. Courant and D. Hilbert, *Methods of Mathematical Physics, Volume I*, John Wiley, New York, (1953).
11. Numerical Algorithms Group Library Mark 16, NAG(U.K.) Ltd, Wilkinson House, Oxford, U.K., (1993).