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Technical note

The method of fundamental solutions for three-dimensional elastostatics problems

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Abstract

We consider the application of the method of fundamental solutions to isotropic elastostatics problems in three space dimensions. The displacements are approximated by linear combinations of the fundamental solutions of the Cauchy–Navier equations of elasticity, which are expressed in terms of sources placed outside the domain of the problem under consideration. The final positions of the sources and the coefficients of the fundamental solutions are determined by enforcing the satisfaction of the boundary conditions in a least squares sense. The applicability of the method is demonstrated on two test problems. The numerical experiments indicate that accurate results can be obtained with relatively few degrees of freedom.

Keywords: Method of fundamental solutions; Elasticity; Isotropic materials

1. Introduction

The method of fundamental solutions (MFS) is a method for the solution of certain elliptic boundary value problems, which may be viewed as an indirect boundary element method. In the MFS, the solution is approximated by a set of fundamental solutions of the governing equations which are expressed in terms of sources located outside the domain of the problem. The unknown coefficients in the linear combination of the fundamental solutions and the final locations of the sources are determined so that the boundary conditions are satisfied in a least squares sense. The method is relatively easy to implement, it is adaptive in the sense that it takes into account sharp changes in the solution and/or in the geometry of the domain and can easily incorporate difficult boundary conditions [10]. A survey of the MFS and related methods over the last thirty years may be found in Ref. [4].

The charge simulation method (CSM), which is a boundary method related to the MFS in which the sources are fixed, has been already used for the solution of two- and three-dimensional elastostatics problems. Burgess and Mahajerin [3], Patterson and Sheikh [9] and Redekop [11] solved two-dimensional elastostatics problems. Redekop and Cheung [12] solved three-dimensional elastostatics problems and Redekop and Thompson [13] employed the CSM for the solution of axisymmetric elastostatics problems. More recently, the MFS has been used for the solution of two-dimensional [2] and axisymmetric [6] elastostatics problems.

The objective of this paper is to formulate the MFS for the solution of three-dimensional isotropic linear elastic problems. The applicability of the method is demonstrated on various test cases. The governing equations and the MFS formulation are discussed in Section 2. The numerical results are presented in Section 3, and the conclusions are summarized in Section 4.

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2. Governing equations and the MFS formulation for isotropic problems

In the absence of body forces, the governing equations of equilibrium for a homogeneous isotropic linear elastic solid are the Cauchy–Navier equations. Using the indicial tensor notation in terms of the displacements u_i, $i = 1,2,3$, the Cauchy–Navier equations, in a bounded three-dimensional domain Ω of the solid, take the dimensionless form $(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} = 0$ [5], where λ and μ are the Lamé elastic constants. These constants can be expressed as $\lambda = vE/((1+v)(1-2v))$ and $\mu = E/2(1+v)$, where E is the modulus of elasticity, and v is Poisson's ratio. In the above equation, summation over repeated subscripts is implied and partial derivatives are denoted by $u_{i,j} = \partial u_i/\partial x_j$. In the linear theory, the strains ε_{ij}, $i,j = 1,2,3$, are related to the displacement gradients by means of $\varepsilon_{ij} = \frac{1}{2}\left((\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)\right)$, and the stresses σ_{ij}, $i,j = 1,2,3$, are given by Hooke's law $\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + 2\mu\varepsilon_{ij}$. The tractions t_j, $j = 1,2,3$ are defined in terms of the stresses as $t_i = \sigma_{ij}n_j$, where n_1, n_2 and n_3 denote the coordinates of the outward normal to the boundary.

In Cartesian coordinates, the Cauchy–Navier equations for the displacements u_1, u_2 and u_3 become

$$\left(\frac{2-2v}{1-2v}\right)\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} + \left(\frac{1}{1-2v}\right)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \left(\frac{1}{1-2v}\right)\frac{\partial^2 u_3}{\partial x_1 \partial x_3} = 0, \tag{1}$$

$$\left(\frac{1}{1-2v}\right)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2} + \left(\frac{2-2v}{1-2v}\right)\frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} + \left(\frac{1}{1-2v}\right)\frac{\partial^2 u_3}{\partial x_2 \partial x_3} = 0, \tag{2}$$

$$\left(\frac{1}{1-2v}\right)\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \left(\frac{1}{1-2v}\right)\frac{\partial^2 u_2}{\partial x_2 \partial x_3} + \frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \left(\frac{2-2v}{1-2v}\right)\frac{\partial^2 u_3}{\partial x_3^2} = 0, \quad \text{in } \Omega. \tag{3}$$

These are subject to the boundary conditions

$$B_i[u_1, u_2, u_3, t_1, t_2, t_3] = f_i \quad \text{on } \Omega, \quad i = 1,2,3, \tag{4}$$

where $\partial\Omega$ is the boundary of Ω, which we shall assume to be piecewise smooth. The operators B_i, $i = 1,2,3$, specify Dirichlet, Neumann or Robin boundary conditions. For a source located at a point Q acting at a point P, the fundamental solutions of the system (1)–(3) are (see, e.g., Refs. [1,5])

$$G_{11}(P,Q) = \frac{1}{16\pi\mu(1-v)}\left[\frac{(3-4v)r_{PQ}^2 + (x_{1_P} - x_{1_Q})^2}{r_{PQ}^3}\right], \tag{5}$$

$$G_{12}(P,Q) = G_{21}(P,Q) = \frac{1}{16\pi\mu(1-v)}\left[\frac{(x_{1_P} - x_{1_Q})(x_{2_P} - x_{2_Q})}{r_{PQ}^3}\right], \tag{6}$$

$$G_{13}(P,Q) = G_{31}(P,Q) = \frac{1}{16\pi\mu(1-v)}\left[\frac{(x_{1_P} - x_{1_Q})(x_{3_P} - x_{3_Q})}{r_{PQ}^3}\right], \tag{7}$$

$$G_{22}(P,Q) = \frac{1}{16\pi\mu(1-v)}\left[\frac{(3-4v)r_{PQ}^2 + (x_{2_P} - x_{2_Q})^2}{r_{PQ}^3}\right], \tag{8}$$

$$G_{23}(P,Q) = G_{32}(P,Q) = \frac{1}{16\pi\mu(1-v)}\left[\frac{(x_{2_P} - x_{2_Q})(x_{3_P} - x_{3_Q})}{r_{PQ}^3}\right], \tag{9}$$

$$G_{33}(P,Q) = \frac{1}{16\pi\mu(1-v)}\left[\frac{(3-4v)r_{PQ}^2 + (x_{3_P} - x_{3_Q})^2}{r_{PQ}^3}\right], \tag{10}$$

where

$$r_{PQ} = \sqrt{(x_{1_P} - x_{1_Q})^2 + (x_{2_P} - x_{2_Q})^2 + (x_{3_P} - x_{3_Q})^2}.$$

The displacements are approximated by linear combinations of fundamental solutions:

$$u_{1_N}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{Q};\ P) = \sum_{j=1}^{N} a_j G_{11}(P,Q_j) + \sum_{j=1}^{N} b_j G_{12}(P,Q_j) + \sum_{j=1}^{N} c_j G_{13}(P,Q_j), \tag{11}$$

$$u_{2_N}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{Q};\ P) = \sum_{j=1}^{N} a_j G_{21}(P,Q_j) + \sum_{j=1}^{N} b_j G_{22}(P,Q_j) + \sum_{j=1}^{N} c_j G_{23}(P,Q_j), \tag{12}$$

$$u_{3_N}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{Q};\ P) = \sum_{j=1}^{N} a_j G_{31}(P,Q_j) + \sum_{j=1}^{N} b_j G_{32}(P,Q_j) + \sum_{j=1}^{N} c_j G_{33}(P,Q_j), \tag{13}$$

and the tractions are approximated accordingly [7]. In the above equations, N is the specified number of sources, $P \in \overline{\Omega} = \Omega \cup \partial\Omega$, and $\mathbf{a} = (a_1, a_2, \ldots, a_N)$, $\mathbf{b} = (b_1, b_2, \ldots, b_N)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_N)$ are vectors containing unknown coefficients. The $3N$-vector $\mathbf{Q}$ contains the coordinates of the sources Q_j, which lie outside $\overline{\Omega}$. A set of points $\{P_i\}_{i=1}^{M}$ is selected on $\partial\Omega$. The coefficients $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and the locations of the sources $\mathbf{Q}$ (a total of $6N$ unknowns) are determined by minimizing the functional

$$\begin{aligned} F(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{Q}) = {} & \sum_{i=1}^{M} \{B_1[u_{1_N}, u_{2_N}, u_{3_N}, t_{1_N}, t_{2_N}, t_{3_N}](\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{Q};P_i) - f_1(P_i)\}^2 \\ & + \sum_{i=1}^{M} \{B_2[u_{1_N}, u_{2_N}, u_{3_N}, t_{1_N}, t_{2_N}, t_{3_N}](\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{Q};P_i) - f_2(P_i)\}^2 \\ & + \sum_{i=1}^{M} \{B_3[u_{1_N}, u_{2_N}, u_{3_N}, t_{1_N}, t_{2_N}, t_{3_N}](\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{Q};P_i) - f_3(P_i)\}^2. \end{aligned} \tag{14}$$

The minimization of the functional F is achieved using the nonlinear least squares package LMDIF from MINPACK [8]. This routine minimizes the sum of squares of m nonlinear functions in n variables using a modified version of the Levenberg–Marquard algorithm and terminates when the specified number of function evaluations is reached. A function evaluation occurs each time there is a call to the subroutine which calculates one of the functions $B_1(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{Q};P_i)$, $B_2(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{Q};P_i)$ and $B_3(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{Q};P_i)$. Explicit formulae for the derivatives required in the evaluation of the approximations of the tractions are given in Ref. [7].

3. Numerical results

3.1. Example 1

We considered the solution of the Cauchy–Navier equations in the cube shown in Fig. 1 with $\Omega = (-1,1)\times(-1,1)\times(-1,1)$ subject to various boundary conditions corresponding to the exact solution $u_1 = x_1$, $u_2 = x_2$ and $u_3 = x_3$. For the elastic constants, we used values appropriate for steel, namely, $\mu = 1.15$ and $v = 0.3$. Similar examples were considered in Ref. [12]. We obtained results for different boundary conditions while varying the number of sources, N, and the number of function evaluations, NFEV. In each case we registered the maximum absolute errors on a uniform $0.1 \times 0.1 \times 0.1$ grid in the MFS approximations, e_1, e_2 and e_3 corresponding to the displacements u_1, u_2 and u_3, respectively. In Table 1, we present results in the case when only Dirichlet boundary conditions are imposed for the displacements and in Table 2, we present results when tractions are prescribed on the three sides S_2, S_4 and S_6 and the displacements are prescribed on the remaining sides S_1, S_3 and S_5. Results for other combinations of boundary conditions may be found in Ref. [7]. In all the cases examined, the accuracy of the approximation improves as the number of degrees of freedom and the number of function evaluations are increased.

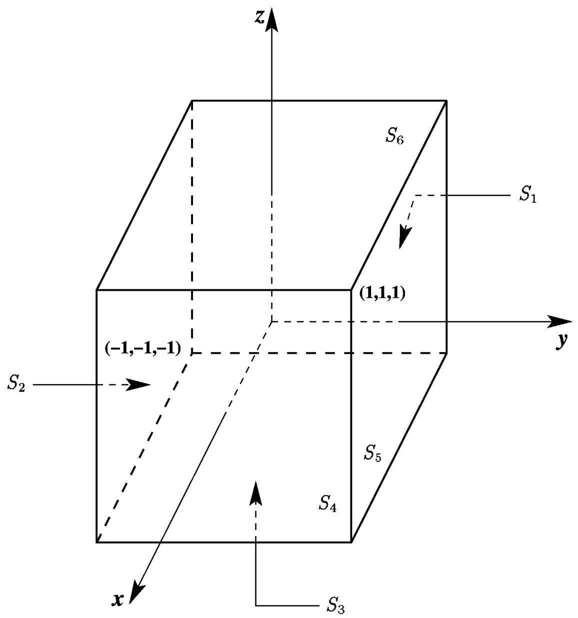

Fig. 1. Geometry of first example problem (cubic domain).

Table 1
Example 1, maximum absolute errors for Dirichlet problem

N	M	NFEV	e_1	e_2	e_3
6	96	1000	0.272-1	0.272-1	0.314-1
		2000	0.762-2	0.762-2	0.809-2
		3000	0.480-2	0.480-2	0.463-2
		4000	0.379-2	0.379-2	0.319-2
		5000	0.312-2	0.312-2	0.319-2
9	150	1000	0.327-1	0.302-1	0.348-1
		2000	0.441-2	0.355-2	0.538-2
		3000	0.220-2	0.186-2	0.244-2
		4000	0.140-2	0.131-2	0.152-2
		5000	0.107-2	0.101-2	0.113-2
12	216	1000	0.101-0	0.122-0	0.784-1
		2000	0.664-2	0.534-2	0.461-2
		3000	0.148-2	0.127-2	0.121-2
		4000	0.659-3	0.579-3	0.573-3
		5000	0.363-3	0.326-3	0.326-3

3.2. Example 2

In order to test the method for a problem with a curved boundary, we also examined the solution of the Cauchy–Navier equations in a cylinder of height equal to 2 and radius equal to 1, shown in Fig. 2. In Table 3, we present the results obtained in the case when Dirichlet boundary conditions corresponding to the exact solution $u_1 = x_1$, $u_2 = x_2$ and $u_3 = x_3$, are imposed everywhere on the boundary (i.e. on S_1, S_2 and S_3). A similar example was considered in [6]. These results are comparable to the ones obtained in Example 1, that is, the accuracy of the approximation improves as the number of degrees of freedom and the number of function evaluations are increased.

Table 2
Example 1, maximum absolute errors when the tractions are prescribed on S_2, S_4 and S_6

N	M	NFEV	e_1	e_2	e_3
6	96	1000	0.148-0	0.124-0	0.167-0
		2000	0.118-0	0.147-0	0.765-1
		3000	0.161-1	0.228-1	0.133-1
		4000	0.660-2	0.935-2	0.498-2
		5000	0.386-2	0.612-2	0.274-2
9	150	1000	0.807-1	0.807-0	0.807-0
		2000	0.124-1	0.124-1	0.124-1
		3000	0.537-2	0.537-2	0.537-2
		4000	0.303-2	0.303-2	0.301-2
		5000	0.208-2	0.208-2	0.207-2
12	216	1000	0.156-0	0.127-0	0.141-0
		2000	0.921-2	0.105-1	0.121-1
		3000	0.160-2	0.135-2	0.147-2
		4000	0.114-2	0.959-3	0.989-3
		5000	0.835-3	0.705-3	0.705-3

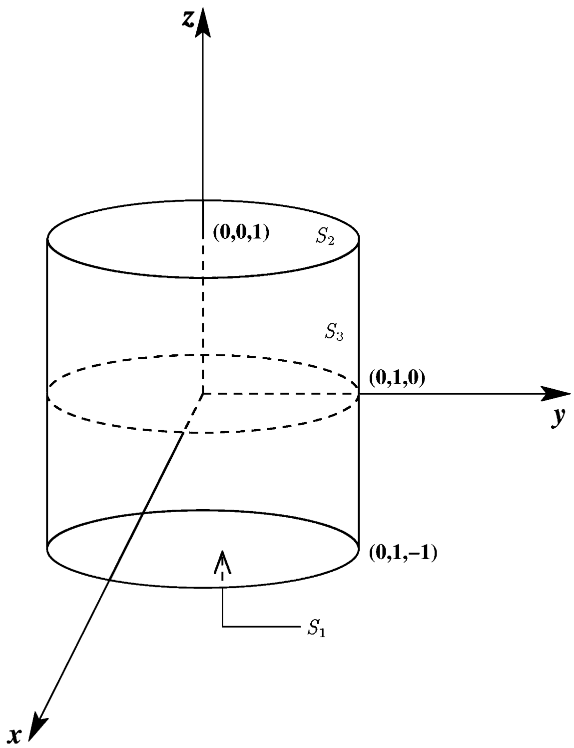

Fig. 2. Geometry of second example problem (cyclindrical domain).

4. Conclusions

In this work, we describe the application of the MFS to three-dimensional problems of steady-state elasticity. The method is very easy to implement, requires little data preparation, and, unlike boundary element methods [5], it avoids potentially troublesome and costly integrations on the boundary. The numerical tests indicate that satisfactory accuracy is obtained with relatively few degrees of freedom.

Table 3
Example 2, maximum absolute errors for Dirichlet problem

N	M	NFEV	e_1	e_2	e_3
6	96	1000	0.744-0	0.782-0	0.750-0
		2000	0.141-0	0.623-0	0.107-0
		3000	0.405-1	0.352-1	0.471-1
		4000	0.139-1	0.141-1	0.156-1
		5000	0.492-2	0.640-2	0.712-2
9	150	1000	0.384-1	0.367-1	0.567-2
		2000	0.763-2	0.617-2	0.538-2
		3000	0.388-2	0.308-2	0.269-2
		4000	0.233-2	0.184-2	0.159-2
		5000	0.161-2	0.137-2	0.120-2
12	216	1000	0.468-0	0.231-0	0.167-0
		2000	0.523-2	0.672-2	0.627-2
		3000	0.198-2	0.247-2	0.228-2
		4000	0.873-3	0.134-2	0.124-2
		5000	0.463-3	0.703-3	0.638-3

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