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# A note on the unbounded creeping flow past a sphere for Newtonian fluids with pressure-dependent viscosity

Kostas D. Housiadas [a,*], Georgios C. Georgiou [b], Roger I. Tanner [c]

[a] Department of Mathematics, University of the Aegean, Karlovassi, 83200 Samos, Greece
[b] Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus
[c] School of Aerospace, Mechanical and Mechatronic Engineering, University of Sydney, Sydney, NSW 2006, Australia

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ABSTRACT

We investigate theoretically isothermal, incompressible, creeping Newtonian flows past a sphere, under the assumption that the shear viscosity is pressure-dependent, varying either linearly or exponentially with pressure. In particular, we consider the three-dimensional flow past a freely rotating neutrally buoyant sphere subject to shear at infinity and the axisymmetric flow past a sedimenting sphere. The method of solution is a regular perturbation scheme with the small parameter being the dimensionless coefficient which appears in the expressions for the shear viscosity. Asymptotic solutions for the pressure and the velocity field are found only for the simple shear case, while no analytical solutions could be found for the sedimentation problem. For the former flow, calculation of the streamlines around the sphere reveals that the fore-and-aft symmetry of the streamlines which is observed in the constant viscosity case breaks down. Even more importantly, the region of the closed streamlines around the sphere is absent. Last, it is revealed that the angular velocity of the sphere is not affected by the dependence of the viscosity on the pressure.



## 1. Introduction

In most isothermal flows of Newtonian liquids, the shear viscosity is commonly assumed to be constant. However, the viscosity of typical liquids begins to increase substantially with pressure when pressures of the order of 1000 atm are reached (Denn, 2008; Rajagopal, Saccomandi, & Vergori, 2012; Renardy, 2003). In certain applications, the effect of the pressure on the viscosity is much larger than that on the mass density, so that compressibility may be neglected but the viscosity pressure dependence needs to be accounted for (Denn, 2008; Goubert, Vermant, Moldenaers, Göttfert, & Ernst, 2001). Hence, the assumption of constant viscosity is valid only at low processing pressures and may introduce error when modeling flows involving high pressures or a large pressure range, e.g. in polymer and food processing, pharmaceutical tablet manufacturing, crude oil and fuel oil pumping, fluid film lubrication, microfluidics, and geophysics (Dealy & Wang, 2013; Le Roux, 2009; Martinez-Boza, Martin-Alfonso, Callegos, & Fernández, 2011; Rajagopal et al., 2012). Due to the growing interest in applications of high pressure chemical and process technologies across a range of engineering fields, flows of fluids with pressure-dependent viscosity as well as techniques for measuring the pressure dependence of the viscosity and the viscosity at high pressure have received increased attention recently (Goubert et al., 2001; Park, Lim, Laun, & Dealy, 2008; Schaschke, 2010). The pressure dependence of viscosity, however, is not only of industrial but also of great fundamental importance.

* Corresponding author.

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Málek and Rajagopal (2007) reviewed different empirical equations proposed in the literature in order to describe experimental observations on the pressure-dependence of the viscosity. Barus (1893) proposed the following formula for the viscosity, $\eta^*$:

$$\eta^* = \eta_0^* \exp\left[\delta^*(p^* - p_0^*)\right] \tag{1}$$

where $\delta^*$ is the pressure-dependence coefficient, assumed to be constant, and $\eta_0^*$ is the viscosity at the reference pressure $p_0^*$. It should be noted that throughout the text a superscript $^*$ denotes a dimensional quantity. Barus (1891) employed the following linear expression:

$$\eta^* = \eta_0^*\left[1 + \delta^*(p^* - p_0^*)\right] \tag{2}$$

which is equivalent to Eq. (1) for small values of $\delta^*$ and/or small pressure differences. As noted by Schaschke (2010), simple models like the above may be used only for simple and small molecules and not for long chain molecules, such as polymers and oil mixtures. Other formulae proposed in the literature, which better fit experimental results for complex fluids, can be found in Málek and Rajagopal (2007).

The pressure-dependence of the viscosity in lubrication (Szeri, 1998), viscometric and other flows has been analyzed mathematically by various investigators (Hron, Málek, & Rajagopal, 2001; Lanzendörfer & Stebel, 2011; Málek & Rajagopal, 2007; Marušić & Pažanin, 2013; Renardy, 2003). Hron et al. (2001) studied various unidirectional and two-dimensional flows of simple fluids with pressure-dependent viscosities and showed that unidirectional flows corresponding to Couette or Poiseuille flow are possible only in special forms of the viscosity. Kalogirou, Poyiadji, and Georgiou (2011) compiled analytical solutions for internal, Poiseuille-type, steady flows. More specifically, these authors studied the unidirectional plane, axisymmetric and annular Poiseuille flows of a Newtonian liquid assuming that the viscosity obeys Eq. (2). Pruša, Srinivasan, and Rajagopal (2012) have recently investigated the role of pressure-dependent viscosity in measurements with falling cylinder viscometers and showed that the error introduced by the application of the classical constant-viscosity formula can be significant for some fluids. They also proposed a heuristic correction to that formula.

So far, however, investigation of external flows with pressure-dependent viscosity are almost absent from the literature. As far as we are aware, no analytical solutions exists, and only some limiting numerical results for the unbounded axisymmetric flow past a sedimenting sphere in a power-law ambient fluid have been presented by Chung and Vaidya (2010). Although in external inertialess flows, one does not expect very large variations of the pressure due to flow, the non-linearity of the governing equations may be adequate to predict unexpected and new flow phenomena.

Of particular interest to the present work is the flow around a sphere, i.e. the simplest flow relevant to falling body type viscometry. More specifically, we have chosen to study the influence of a pressure-dependent shear viscosity on the steady, creeping, incompressible flow of a Newtonian liquid of mass density $\rho_f^*$ past a rigid sphere of radius $R^*$ and mass density $\rho_s^*$. Two different cases are investigated. In the first case, the mass densities of the fluid and the sphere are assumed to be equal, i.e. the sphere is neutrally buoyant, and shear is applied far from the sphere. In the second case, $\rho_s^* > \rho_f^*$ but no shear is applied, and thus the sphere sediments with a constant terminal velocity.

For a neutrally buoyant sphere under the influence of simple shear flow imposed far from the sphere, it is known that non-linear effects, such as inertia or viscoelasticity, break the fore-and-aft symmetric configuration of the streamlines around the sphere, in the plane which shear is applied. Indeed, Lin, Peery, and Schowalter (1970) and Subramanian and Koch (2007), among others, have shown that the inclusion of the inertia terms into the governing equations destroys the symmetry of the orbits of the fluid elements around the sphere. This may have important consequences for the heat transfer around the spherical particle, as well for the bulk properties of suspensions of particulates (Subramanian & Koch, 2007). D'Avino et al. (2008) and Housiadas and Tanner (2011) considered the case where the matrix fluid is a viscoelastic fluid and showed that the fore-and-aft symmetry of the streamlines also then breaks down. In the present work, it is shown that an alternative cause of the destruction of the streamline symmetry around a spherical particle is the non-linearity introduced by the pressure dependence of the viscosity of the matrix fluid.

Recently, is has been demonstrated that for internal, fully developed, laminar flows in straight channels and circular tubes the perturbation solution up to fourth order in $\delta$ (for the definition of the dimensionless coefficient $\delta$ see the subsequent section) is an excellent approximation of the full, analytical solution (Poyiadji, Housiadas, Kaouri, & Georgiou, 2015). For the problems under consideration, and for typical flow conditions, the dimensionless pressure-viscosity coefficient is a small number, i.e. $\delta \ll 1$. Since the full, non-linear, governing equations cannot be solved analytically, we employed a regular perturbation scheme with the small number being the $\delta$ coefficient.

The rest of the paper is organized as follows. In Section 2, the assumptions, governing equations and boundary conditions are presented in dimensionless form. In Section 3, the solution procedure and the analytical solution up to first order is presented for the simple shear case. In the same section, the streamlines around the sphere are calculated and discussed. The main conclusions are summarized in Section 4. Finally, in Appendix A, it is shown that no analytical solution can be found with the proposed method for the two-dimensional flow past a sedimenting sphere.

## 2. Problem definition

The steady, creeping, isothermal flow around a sphere is considered. The ambient fluid is assumed to be Newtonian with constant mass density $\rho_f^*$, and a variable shear viscosity $\eta^*$, given by either one of Eqs. (1) and (2). A fixed spherical coordinate system $(r, \theta, \phi)$ with the origin at the centre of the sphere is used; the unit vectors are denoted by $\mathbf{e_r}$, $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$. A schematic representation of the flow is illustrated in the left panel in Fig. 1, where a Cartesian coordinate system $(x,y,z)$ is also shown for convenience. The $z$-axis is in the direction of gravity and shear is applied in the $xy$ plane. The Cartesian and spherical coordinates are connected through the standard relations $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$, and $z = r\cos(\theta)$.

The radius of the sphere, $R^*$, is used to scale all lengths, and the quantity $R^*\dot{\gamma}^*$ is used to scale the velocity vector $\mathbf{v}^*$. The viscosity is made dimensionless by $\eta_0^*$ and the pressure difference $p^* - p_0^*$ is scaled by $\eta_0^*\dot{\gamma}^*$. With these characteristic scales and the aforementioned assumptions, the dimensionless governing equations in absence of any other external forces and torques are written as follows:

$$\nabla \cdot \mathbf{v} = 0 \tag{3}$$

$$-\nabla p + \eta\nabla^2\mathbf{v} + \nabla\eta \cdot \dot{\boldsymbol{\gamma}} = \mathbf{0} \tag{4}$$

where $\dot{\boldsymbol{\gamma}} = \nabla\mathbf{v} + (\nabla\mathbf{v})^T$ is the rate of deformation tensor, and $T$ denotes the transpose of a tensor. The dimensionless equations for the shear viscosity as a function of pressure are

$$\eta = 1 + \delta p \tag{5}$$

and

$$\eta = \exp(\delta p) \tag{6}$$

for the linear and the exponential (Barus) cases, respectively, where

$$\delta \equiv \delta^*\eta_0^*\dot{\gamma}^* \tag{7}$$

The dimensionless flow domain is $\{1 \leqslant r < \infty,\ 0 < \theta < \pi,\ 0 \leqslant \phi < 2\pi\}$.

The no-slip and no-penetration conditions are assumed to hold at the surface of the sphere:

$$\mathbf{v} = \Omega\sin(\theta)\mathbf{e}_\theta \quad \text{at} \quad r = 1 \tag{8}$$

where $\Omega$ is the unknown angular velocity of the sphere, which is evaluated by means of the torque-free-condition:

$$\int_{r=1} \mathbf{e_r} \times \{(-p\mathbf{I} + \eta\dot{\boldsymbol{\gamma}}) \cdot \mathbf{e_r}\} dS = \mathbf{0} \tag{9}$$

where $\mathbf{I}$ is the unit tensor.

Far from the sphere the velocity approaches a steady terminal value of the sphere and the pressure becomes equal to the datum pressure which can be taken to be zero:

$$\mathbf{v} \to \mathbf{v}^{(\infty)} = y\mathbf{e_x}, \quad p \to p^{(\infty)} = 0, \quad \text{at} \quad r \to \infty \tag{10}$$

In spherical coordinates, $y = r\sin(\theta)\sin(\phi)$ and $\mathbf{e_x} = \sin(\theta)\cos(\phi)\mathbf{e_r} + \cos(\theta)\cos(\phi)\mathbf{e}_\theta - \sin(\phi)\mathbf{e}_\varphi$; thus the velocity profile far from the sphere, $\mathbf{v}^{(\infty)}$, becomes:

$$\mathbf{v}^{(\infty)} = \left\{\frac{r}{4}(1 - \cos(2\theta))\sin(2\phi)\right\}\mathbf{e_r} + \left\{\frac{r}{4}\sin(2\theta)\sin(2\phi)\right\}\mathbf{e}_\theta + \left\{-\frac{r}{2}\sin(\theta)(1 - \cos(2\phi))\right\}\mathbf{e}_\varphi \tag{11}$$

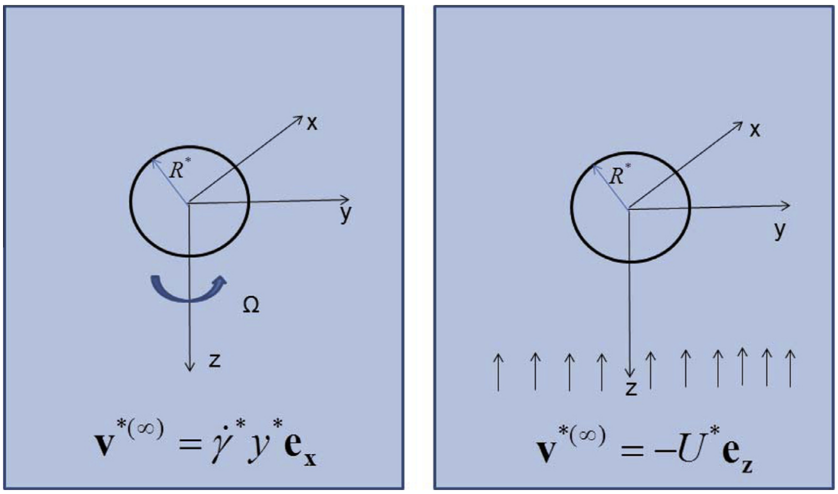


**Fig. 1.** Flow configuration and coordinate system. Left: A neutrally buoyant, freely rotating sphere under simple shear at infinity. Right: Sedimentation of a sphere in a stream with uniform velocity.

## 3. Solution and discussion

Since the dimensionless pressure-viscosity coefficient is a small number, i.e. $\delta \ll 1$, a regular perturbation scheme is employed as follows:

$$X = X_0 + \delta X_1 + \delta^2 X_2 + \delta^3 X_3 + \delta^4 X_4 + O(\delta^5), \quad X \in \{\mathbf{v}, p, \eta, \Omega\} \tag{12}$$

Substituting expression (12) into the governing equations and collecting terms of the same order of magnitude, result in a sequence of sets of partial differential equations. The zero-order problem corresponds to the creeping flow for a Newtonian fluid with constant viscosity, i.e. $\eta_0 = 1$:

$$\nabla \cdot \mathbf{v}_0 = 0, \quad -\nabla p_0 + \nabla^2 \mathbf{v}_0 = \mathbf{0}, \quad \nabla^2 p_0 = 0 \tag{13}$$

$$v_{r,0} = v_{\theta,0} = 0, \, v_{\phi,0} = \Omega_0 \sin(\theta), \quad \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \dot{\gamma}_{r\phi,0} \sin^2(\theta) d\theta d\phi = 0 \text{ at } r = 1 \tag{14}$$

$$\mathbf{v}_0(r \to \infty) = y\mathbf{e_x}, \quad p_0(r \to \infty) = 0 \tag{15}$$

The solution of this (linear) problem can be derived with various methods (see, e.g., Leal, 2007):

$$\left.\begin{array}{l} p_0 = -\frac{5}{4r^3}(1 - \cos(2\theta)) \sin(2\phi) \\ v_{r,0} = (1 - \cos(2\theta)) \sin(2\phi) \frac{r}{4}\left(1 - \frac{5}{2r^3} + \frac{3}{2r^5}\right) \\ v_{\theta,0} = \sin(2\theta) \sin(2\varphi) \frac{r}{4}\left(1 - \frac{1}{r^5}\right) \\ v_{\phi,0} = -\frac{r}{2} \sin(\theta)\left\{1 - \cos(2\phi)\left(1 - \frac{1}{r^5}\right)\right\} \end{array}\right\} \tag{16}$$

The higher-order problems in the small parameter, $\delta$, are as follows:

$$\nabla \cdot \mathbf{v}_j = 0, \quad -\nabla p_j + \nabla^2 \mathbf{v}_j + \mathbf{f}_j = \mathbf{0}, \quad \nabla^2 p_j = \nabla \cdot \mathbf{f}_j \tag{17}$$

$$v_{r,j} = v_{\theta,j} = 0, \quad v_{\phi,j} = \Omega_j \sin(\theta), \quad \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (\eta \dot{\gamma}_{r\varphi})_j \sin^2(\theta) d\theta d\phi = 0 \text{ at } r = 1 \tag{18}$$

$$\mathbf{v}_j = p_j = 0 \text{ at } \quad r \to \infty \tag{19}$$

where $\mathbf{f}_j$ is a vector which depends on all the lower-order solutions for the velocity and the pressure, i.e. $\mathbf{f}_j = \mathbf{f}_j\ (\mathbf{v}_k, p_k)$, $k = 0, 1, \ldots j - 1$. In particular:

$$\mathbf{f}_1 = p_0 \nabla^2 \mathbf{v}_0 + \nabla p_0 \cdot \dot{\gamma}_0$$

$$\mathbf{f}_2 = p_0\left(\zeta \nabla p_0 \cdot \dot{\gamma}_0 + \nabla^2 \mathbf{v}_1\right) + \left(\zeta \frac{p_0^2}{2} + p_1\right) \nabla^2 \mathbf{v}_0 + \nabla p_0 \cdot \dot{\gamma}_1 + \nabla p_1 \cdot \dot{\gamma}_0$$

$$\begin{aligned} \mathbf{f}_3 = {} & p_0\left[\zeta(\nabla p_0 \cdot \dot{\gamma}_1 + \nabla p_1 \cdot \dot{\gamma}_0) + \nabla^2 \mathbf{v}_2\right] + \left(\zeta \frac{p_0^2}{2} + p_1\right)\left(\nabla p_0 \cdot \dot{\gamma}_0 + \nabla^2 \mathbf{v}_1\right) + \left[\zeta\left(\frac{p_0^3}{6} + p_0 p_1\right) + p_2\right] \nabla^2 \mathbf{v}_0 \\ & + \nabla p_0 \cdot \dot{\gamma}_2 + \nabla p_1 \cdot \dot{\gamma}_1 + \nabla p_2 \cdot \dot{\gamma}_0 \end{aligned}$$

$$\begin{aligned} \mathbf{f}_4 = {} & p_0\left[\zeta(\nabla p_0 \cdot \dot{\gamma}_2 + \nabla p_1 \cdot \dot{\gamma}_1 + \nabla p_2 \cdot \dot{\gamma}_0) + \nabla^2 \mathbf{v}_3\right] + \left(\zeta \frac{p_0^2}{2} + p_1\right)\left(\nabla p_0 \cdot \dot{\gamma}_1 + \nabla p_1 \cdot \dot{\gamma}_0 + \nabla^2 \mathbf{v}_2\right) \\ & + \left[\zeta\left(\frac{p_0^3}{6} + p_0 p_1\right) + p_2\right]\left(\nabla p_1 \cdot \dot{\gamma}_0 + \nabla^2 \mathbf{v}_1\right) + \left[\zeta\left(\frac{p_0^4}{24} + \frac{p_0^2 p_1 + p_1^2}{2} + p_0 p_2\right) + p_3\right] \nabla^2 \mathbf{v}_0 \\ & + \nabla p_0 \cdot \dot{\gamma}_3 + \nabla p_1 \cdot \dot{\gamma}_2 + \nabla p_2 \cdot \dot{\gamma}_1 + \nabla p_3 \cdot \dot{\gamma}_0 \end{aligned}$$

In the above expressions, $\zeta = 0$ with the linear and $\zeta = 1$ with the exponential formula for the viscosity.

In order to find the solutions for the linear problems given in Eqs. (17)–(19), a solution procedure which is very similar to that previously described by Housiadas and Tanner (2014) is followed. In particular, the primary flow variables in the governing equations at order $O(\delta^j)$, $j = 1, 2, 3, 4$ are assumed to be as follows:

$$X_j = \begin{cases} \displaystyle\sum_{m=0}^{M_j/2} \sum_{n=0}^{M_j} [X_j(r)]_n^{(m)} e^{2im\phi} \cos(n\theta), & X = v_r, p \\ \displaystyle\sum_{m=0}^{M_j} \sum_{n=1}^{M_j} [X_j(r)]_n^{(m)} e^{2im\phi} \sin(n\theta), & X = v_\theta, v_\phi \end{cases} \tag{20}$$

where $M_j = 2(j + 1)$, and $[X_j(r)]_n^{(m)}$ are functions of the radial distance, $r$, that have to be determined. Expression (20), which is valid even for $j = 0$, is substituted in the governing equations, and then the following steps are performed for each wavenumber in the $\phi$-angle, $m$:

(i) Solve the continuity equation, i.e. the first one in (17), to find $v_{\phi,j}$.
(ii) Solve the Poisson equation, i.e. the third one in (17), to find $p_j$.
(iii) Solve the radial component of the momentum equation, i.e. the second one in (17), to find $v_{r,j}$.
(iv) Solve the azimuthal component of the momentum equation, i.e. the second one in (17), to find $v_{\theta,j}$.

In step (i) the azimuthal component of the velocity is found algebraically, while in steps (ii), (iii) and (iv) the differential equations are solved together with the appropriate boundary conditions (18) and (19). Following this procedure, the solutions to the first-, second- and third-order problems are calculated and only the zero Fourier mode in the $\phi$-angle for the fourth-order problem (which is the only Fourier mode that is required for the calculation of the angular velocity of the sphere).

The analytical solution at first order in $\delta$ is the same for both the linear and exponential formulas for the viscosity, and has the following form:

$$\left.\begin{aligned}
p_1 &= p_1^{[0]}(r,\theta) + p_1^{[4]}(r,\theta)\cos(4\phi)\\
v_{r1} &= v_{r1}^{[0]}(r,\theta) + v_{r1}^{[4]}(r,\theta)\cos(4\phi)\\
v_{\theta 1} &= v_{\theta 1}^{[0]}(r,\theta) + v_{\theta 1}^{[4]}(r,\theta)\cos(4\phi)\\
v_{\phi 1} &= v_{\phi 1}^{[4]}(r,\theta)\sin(4\phi)
\end{aligned}\right\} \tag{21}$$

where $[X]^{(m)}$ denotes the $m$-Fourier mode of $X$ in the $\phi$-angle which depends on $r$ and $\theta$:

$$\begin{aligned}
v_{r1}^{[0]} &= \frac{(1-r)^2}{21504r^6}\{-5(-243+212r(2+r)) - 36[25(-3+4r(2+r))]\cos(2\theta) + 525[9+4r(2+r)]\cos(4\theta)\}\\
v_{r1}^{[4]} &= -\frac{25(r-1)^2(9+8r+4r^2)\sin^4(\theta)}{128r^6}\\
v_{\theta 1}^{[0]} &= \frac{(r-1)((90-785r)+105(6+r)\cos(2\theta))\sin(2\theta)}{1792r^6}\\
v_{\theta 1}^{[4]} &= \frac{15(1-r)(6+r)\cos(\theta)\sin^3(\theta)}{64r^6}\\
v_{\phi 1}^{[4]} &= \frac{15(1-r)(6+r)\cos(\theta)\sin^3(\theta)}{64r^6}\\
p_1^{[0]} &= \frac{19740+r^2(-12600-567r+760r^3)}{10752r^8} + \frac{5(-364+r^2(280-21r+24r^3))}{10752r^8}\cos(2\theta)\\
&\quad + \frac{105(20+r^2(-40-21r+40r^3))}{10752r^8}\cos(4\theta)\\
p_1^{[4]} &= \frac{5(-20+40r^2+21r^3-40r^5)\sin^4(\theta)}{64r^8}
\end{aligned}$$

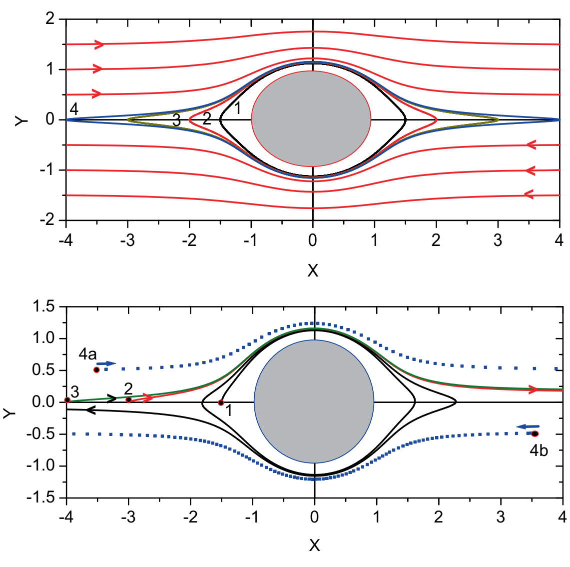


**Fig. 2.** Streamlines on the shear plane for a neutrally buoyant sphere for $\delta = 0$ (top) and $\delta = 0.1$ (bottom).

The analytical solutions at higher orders are too long to be given here. They can be provided however upon request to the authors.

A main quantity of interest is the angular velocity of the sphere. From Eq. (17) one gets $\Omega_0 = -1/2$, while from the higher order solutions we find $\Omega_j = 0,\ j = 1,2,3,4$. Therefore, the rotation of the sphere is exclusively due to the rotation of the ambient fluid at infinity and is not affected by the viscosity pressure dependence (up to fourth order in $\delta$ for both the linear and the exponential cases).

It is also worthy to investigate the effect of the pressure-dependent viscosity on the streamlines. On the $xy$-shear plane, the streamlines are obtained by solving numerically the following initial value problem:

$$\begin{aligned} &\frac{dx}{dt} = v_x \equiv \cos(\phi)v_r(r,\pi/2,\phi) - \sin(\phi)v_\phi(r,\pi/2,\phi) \\ &\frac{dy}{dt} = v_y \equiv \sin(\phi)v_r(r,\pi/2,\phi) + \cos(\phi)v_\phi(r,\pi/2,\phi) \\ &x(0) = x_0, \quad y(0) = y_0 \end{aligned} \tag{22}$$

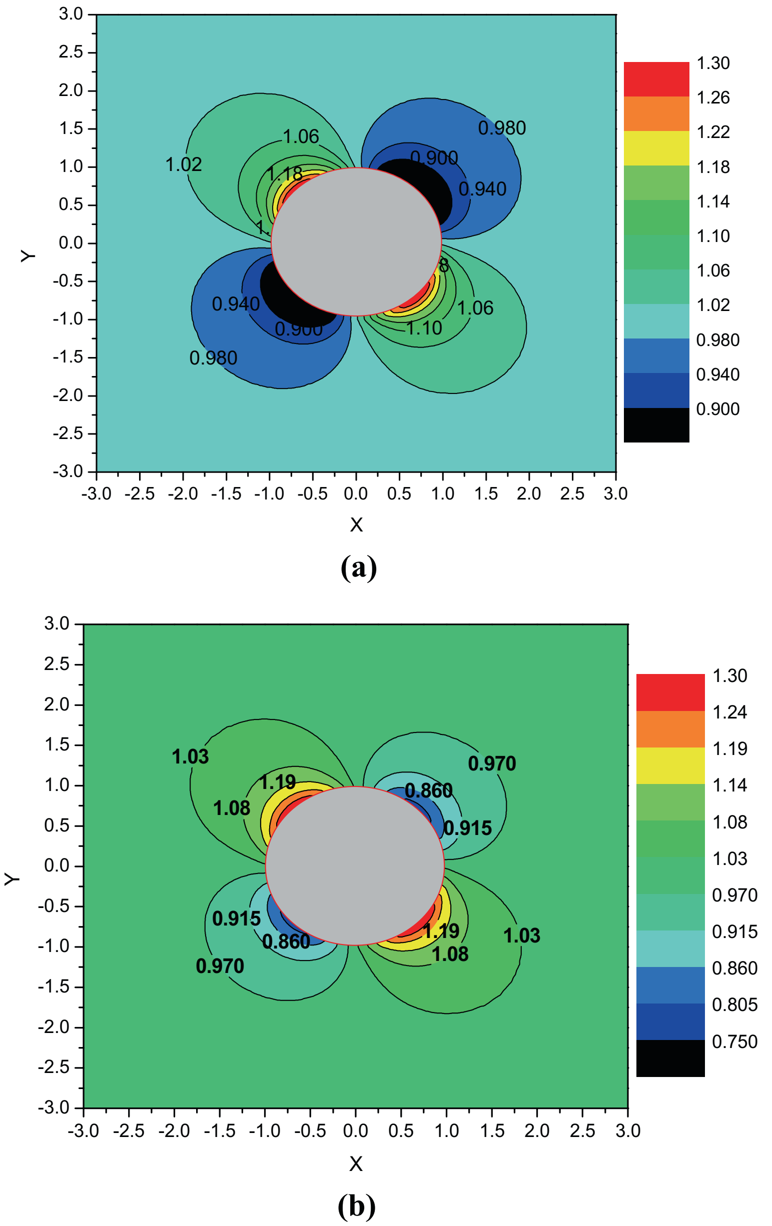


**Fig. 3.** Viscosity (a), and pressure (b) contours on the shear plane, $x-y$, for a neutrally buoyant sphere and a fluid which follows the Barus formula with $\delta = 0.1$.

where $(x_0, y_0)$ is the initial position of any massless fluid element at an arbitrary inception of time ($t = 0$), $v_r(r, \pi/2, \phi)$ and $v_\phi(r, \pi/2, \phi)$ are constructed up to $O(\delta^3)$, and $r = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1}(x, y)$. The integration of Eq. (22) is done using a 4th–5th order accurate Runge–Kutta–Fehlberg method with an error control tolerance $10^{-14}$ in order to achieve the maximum numerical accuracy. The results for constant viscosity ($\delta = 0$) are shown in Fig. 2(a) (see also Leal, 2007, Housiadas & Tanner, 2011). The fore-and-aft and up-and-down symmetry is clearly seen. Also, the streamlines away from the sphere are open, which means that fluid elements at initial positions for which the distance from the y-axis is larger than a critical distance $y_c = y_c(x)$ are only slightly affected by the presence of the sphere. However, fluid elements initially at $y < y_c$ are trapped close to the sphere. For instance, a fluid element which starts at a point $(-1.5,0)$ follows the closed black trajectory labeled with an 1. Similarly, fluid elements which starting positions $(-2,0)$, $(-3,0)$ and $(-4,0)$ follow the trajectories labeled with 2, 3 and 4, respectively.

However, the fore-aft symmetry breaks in the cases for which $\delta > 0$. For instance, for a fluid following the Barus law with $\delta = 0.1$, a fluid element at an initial position $(-1.5,0)$ follows the trajectory labeled 1, ending up at minus infinity, as seen in Fig. 2(b). A simpler trajectory is observed for initial positions $(-3,0)$ and $(-4,0)$, but eventually the fluid elements reach plus infinity. Therefore, it appears that any kind of non-linearity which is caused either by inertia (Lin et al., 1970; Subramanian & Koch, 2006), or by viscoelasticity (D'Avino et al., 2008, Housiadas & Tanner, 2011), or due to variations of the viscosity on the pressure, destroys the fore-and-aft symmetry. Even more importantly, there exists no region of closed streamlines around the sphere. Indeed, all calculations showed that all fluid elements no matter close to the sphere are, and irrespectively of the magnitude of $\delta$, eventually will reach plus or minus infinity. Of course, the required time of course depends on $\delta$; the larger $\delta$ the faster the transition of the fluid elements away from the sphere. It is also seen in Fig. 2(b) that as the distance from the x-axis increases (see for instance the trajectories for fluid elements which start at positions $(-3.5,0.5)$ and $(3.5,-0.5)$, labeled as 4a and 4b, respectively), the influence of the viscosity variations due to pressure diminishes, and gradually the streamlines tend to those for constant viscosity.

The viscosity and pressure contours for a neutrally buoyant sphere and a fluid which follows the Barus formula with $\delta = 0.1$ are illustrated in Fig. 3. In particular, at $\theta = \pi/2$, i.e. at the shear plane $(x, y)$, on which the third component of the velocity vector ($v_z = 0$ in the Cartesian coordinates, or $v_\theta = 0$ in the spherical coordinates) is zero, the viscosity contours are seen in Fig. 3(a); one is reminded here, that $x$ corresponds to the main flow direction, and $y$ corresponds to the gradient direction. It is seen that viscosity contours with values larger than unity are observed in the second (lower-right) and fourth (upper-left) quadrature, while contours with values less that unity are observed in the first (upper-right) and third (lower-left) quadrature. Thus, the viscosity is minimized along the first principal axis of the rate of strain tensor (i.e. at $x = y$), and is maximized along the second principal axis (i.e. at $x = -y$). This is a consequence of the pressure distribution around the sphere. Indeed, Fig. 3(b) shows that the pressure contours are qualitatively the same with the viscosity contours, namely larger than unity in the second and fourth quadrature and less than unity is the first and the third one.

The behavior predicted in Fig. 3 may be important for complex fluids like non-colloidal hard-sphere suspensions. A region of lower viscosity in the vicinity of one of the two main principal axes can cause the ambient liquid to flow faster in that region, and slower in the vicinity of the other principal axis, resulting in alignment of the particles in the suspension. Consequently, anisotropic microstructure of the particle distribution in the suspending fluid can be observed; the anisotropic microstructure is known to be responsible for non-zero values for the bulk (average) first- and second- normal stress differences, even in the case of a Newtonian matrix fluid.

## 4. Conclusions

We have studied the unbounded creeping flow past a sedimenting sphere for which the ambient fluid is Newtonian with shear viscosity depending either linearly or exponentially on the total pressure. The governing equations are solved analytically using a regular perturbation scheme with the small parameter being the dimensionless viscosity-pressure coefficient. It is shown that a solution exists only for the simple (three-dimensional) shear case, while no solution could be found for the sedimentation (axisymmetric) case.

The analytical solution reveals that for a neutrally buoyant sphere, the non-linearity of the governing equations, due to the pressure-dependence of the viscosity, destroys the fore-and-aft symmetry of the streamlines around the sphere which are observed in the constant viscosity, inertialess case. Moreover, the closed region of streamlines is eliminated. These features are of importance in the rheology of particulate suspensions, in heat transfer around spherical particles, and in high pressure viscosity measurements with falling body type viscometers (Pruša et al., 2012; Schaschke, 2010; Subramanian & Koch, 2006). Last, the angular velocity of the sphere remains the same as for the constant viscosity, inertialess case.

As explained in Appendix A, it is not possible to derive analytical solutions for the two-dimensional flow past a sedimenting sphere, using the technique described above.

## Appendix A. The simple sedimentation problem (i.e. in absence of shear)

We investigate here the existence of solutions for the simple sedimentation problem (in absence of shear); the flow configuration and the coordinate system are illustrated on the right panel in Fig. 1. In this case, the mass density of the sphere $\rho_s^*$ is greater than the mass density of the fluid, and thus the sphere sediments with a constant velocity $U^*$. Using $U^*$ to scale the velocity vector, $R^*$ to scale all distances, and $\eta_0^* U^*/R^*$ to scale the pressure, the dimensionless governing equations are the continuity equation (Eq. (3)) and the momentum balance:

$$-\nabla p + \eta\nabla^2\mathbf{v} + \nabla\eta\cdot\dot{\gamma} + St\mathbf{e_z} = \mathbf{0}, \tag{A1}$$

where $St \equiv \left(\rho_f^* g^* R^{*2}\right)/(\eta_0^* U^*)$ is the Stokes number. By means of the modified pressure $P \equiv p - Stz$, Eq. (A1) takes the following form:

$$-\nabla P + \eta\nabla^2\mathbf{v} + \nabla\eta\cdot\dot{\gamma} = \mathbf{0} \tag{A2}$$

The dimensionless equation for the shear viscosity is $\eta = 1 + \varepsilon z + \delta P$ in the linear case and $\eta = \exp(\varepsilon z + \delta P)$ in the exponential case (Barus formula), where $\varepsilon \equiv St\delta = \delta^*\rho_f^* g^* R^*$. The boundary conditions for this problem are $\mathbf{v}(r = 1) = \mathbf{0}$, $\mathbf{v}(r \to \infty) = -\mathbf{e_z}$, $P(r \to \infty) = 0$.

Following the perturbation scheme, described in Section 3, in terms of $\delta$, the zero-order equations $O(\delta^0)$ are

$$\nabla\cdot\mathbf{v}_0 = 0, \quad -\nabla P_0 + \eta_0\nabla^2\mathbf{v}_0 + \varepsilon\eta^\zeta\mathbf{e_z}\cdot\dot{\gamma}_0 = \mathbf{0}, \quad -\nabla^2 P_0 + 2\varepsilon\mathbf{e_z}\cdot\nabla^2\mathbf{v}_0 = 0 \tag{A3}$$

The boundary conditions are $\mathbf{v}_0(r = 1) = \mathbf{0}$, $\mathbf{v}_0\ (r \to \infty) = -\mathbf{e_z}$, $P_0(r \to \infty) = 0$. In Eq. (A3), the zero-order viscosity is $\eta_0 = 1 + \varepsilon z$ and $\eta = \exp(\varepsilon z)$ for the linear ($\zeta = 0$) and exponential ($\zeta = 1$) cases, respectively. These lowest-order expressions actually reflect the effect of the hydrostatic pressure on the shear viscosity of the fluid. In Eq. (A3), $\nabla\eta_0 = \varepsilon\eta^\zeta\mathbf{e_z}$, $\nabla\nabla\eta_0 = \mathbf{e_z}\mathbf{e_z}\varepsilon^2\eta^\zeta\zeta$ have been taken into account. Obviously, for the linear case $\nabla\eta_0 = \varepsilon\mathbf{e_z}$, $\nabla\nabla\eta_0 = \mathbf{0}$, while for the exponential case $\nabla\eta_0 = \varepsilon e^{\varepsilon z}\mathbf{e_z}$, $\nabla\nabla\eta_0 = \varepsilon^2 e^{\varepsilon z}\mathbf{e_z}\mathbf{e_z}$.

We explored a variety of methods to solve Eq. (A3) analytically. However, we could not find a solution. A further attempt to consider a second perturbation scheme with the small parameter being $\varepsilon$, did not work as well. In particular, if we assume that:

$$X_0 = X_{0,0} + \varepsilon X_{0,1} + \varepsilon^2 X_{0,2} + O(\varepsilon^3), \quad X_0 = \mathbf{v}_0, P_0, \eta_0 \tag{A4}$$

and substitute in Eq. (A3) and the boundary conditions, the $O(\varepsilon^0)$ problem is the following:

$$\nabla\cdot\mathbf{v}_{0,0} = 0, \quad -\nabla P_{0,0} + \eta_{0,0}\nabla^2\mathbf{v}_{0,0} = \mathbf{0}, \quad \nabla^2 P_{0,0} = 0 \tag{A5}$$

$$\mathbf{v}_{0,0}(r = 1) = \mathbf{0}, \quad \mathbf{v}_{0,0}(r \to \infty) = -\mathbf{e_z}, \quad P_{0,0}(r \to \infty) = 0 \tag{A6}$$

where $\eta_{0,0} = 1$. The solution of Eqs. (A5) and (A6) (with $v_{\phi,0} = 0$ and no-dependence on the azimuthal angle $\phi$) is the well-known two-dimensional solution of steady creeping Newtonian flow past a sedimenting sphere (Leal, 2007):

$$\left.\begin{array}{l} P_0 = p_0 - St\ r\ \cos(\theta) = \frac{3}{2r^2}\cos(\theta) \\ v_{r,0} = \left(-1 + \frac{3}{2r} - \frac{1}{2r^3}\right)\cos(\theta) \\ v_{\theta,0} = \left(1 - \frac{3}{4r} - \frac{1}{4r^3}\right)\sin(\theta) \end{array}\right\} \tag{A7}$$

The $O(\varepsilon^1)$ governing equations and accompanying boundary conditions are:

$$\nabla\cdot\mathbf{v}_{0,1} = 0, \quad -\nabla P_{0,1} + \nabla^2\mathbf{v}_{0,1} + z\nabla P_{0,0} + \mathbf{e_z}\cdot\dot{\gamma}_{0,0} = \mathbf{0}, \quad -\nabla^2 P_{0,1} + 2\mathbf{e_z}\cdot\nabla P_{0,0} = 0 \tag{A8}$$

$$\mathbf{v}_{0,1}(r = 1) = \mathbf{v}_{0,1}(r \to \infty) = \mathbf{0}, \quad P_{0,1}(r \to \infty) = 0 \tag{A9}$$

In fact, Eqs. (A8) and (A9) are the $O(\delta^0\varepsilon^1)$ governing equations. However, there is no solution of these equations since not all of the required (homogeneous) boundary conditions can be satisfied.

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