
Volume 165, issues 11–12, June 2010 ISSN 0377-0257
ELSEVIER
Journal of Non-Newtonian Fluid Mechanics
On-line Access via:
www.elsevier.com/locate/jnnfm

J. Non-Newtonian Fluid Mech. 165 (2010) 676–680

Contents lists available at ScienceDirect

Journal of Non-Newtonian Fluid Mechanics

journal homepage: www.elsevier.com/locate/jnnfm

Short communication

Perturbation solution of the compressible annular Poiseuille flow of a viscous fluid

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ARTICLE INFO

Article history:
Received 17 July 2009
Received in revised form 16 January 2010
Accepted 26 February 2010

Keywords:
Newtonian liquid
Compressible flow
Compressibility
Poiseuille flow
Annulus

ABSTRACT

The isothermal annular Poiseuille flow of a weakly compressible Newtonian liquid with constant shear and bulk viscosities is considered. A linear equation of state is assumed and a perturbation analysis in terms of the primary flow variables is performed up to the first order using the isothermal compressibility as the perturbation parameter. The effects of compressibility, the bulk viscosity, the radii ratio, the aspect ratio, and the Reynolds number on the velocity and pressure fields are studied.

1. Introduction

The present paper is a continuation of our previous work [1] where perturbation solutions have been derived for both the planar and round compressible Poiseuille Newtonian flows, with compressibility serving as the perturbation parameter. The objective here is to derive the perturbation solution for the annular Poiseuille flow of a compressible Newtonian fluid and to investigate the effects of compressibility, the bulk viscosity, the aspect ratio and the Reynolds number on the velocity and pressure fields.

The paper is organized as follows. In Section 2, the governing equations and boundary conditions are presented and nondimensionalized. In Section 3, the perturbation solution is derived only up to first order, due to the fact that the solution in the annular geometry is much more complicated than in the planar and axisymmetric Poiseuille flow. In Section 4, the effects of the various parameters involved are presented and discussed. Finally, Section 5 summarizes the conclusions.

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doi:10.1016/j.jnnfm.2010.02.022

2. Governing equations

In this work, the fluid density, ρ, is assumed to obey a linear equation of state:

$$\rho = \rho_0[1 + \beta(P - P_0)] \tag{1}$$

where P is the pressure, β is the isothermal compressibility, and ρ_0 is the density at a reference pressure P_0. The constitutive equation of a compressible Newtonian fluid is

$$\boldsymbol{\tau} = \eta[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T] + \left(\eta_1 - \frac{2}{3}\eta\right) \nabla \cdot \boldsymbol{u}\boldsymbol{I} \tag{2}$$

where $\boldsymbol{\tau}$ is the viscous stress tensor, $\mathbf{u}$ is the velocity vector, $\nabla\mathbf{u}$ is the velocity gradient tensor, $\mathbf{I}$ is the unit tensor, η denotes the viscosity, and η_1 is the bulk (or dilatational) viscosity. In the present work, both η and η_1 are assumed to be constant, i.e. independent of pressure. The bulk viscosity, η_1, which is very often neglected, is identically zero only for mono-atomic gases at low density and becomes important in polyatomic gases and in liquids [2,3].

We consider the isothermal, steady, axisymmetric annular Poiseuille flow of a weakly compressible Newtonian fluid under zero gravity and no slip at the walls. The inner and outer radii of the annulus are respectively κR and R, with $0 < \kappa < 1$ and its length is L. To nondimensionalize the governing equations, we scale z by L, r by R, ρ by ρ_0, the axial velocity u_z by

$$U = \frac{\dot{M}}{\pi \rho_0 R^2}$$

where $\dot{M}$ is the mass flow rate, the transverse velocity u_r by UR/L, and the pressure by $A\eta LU/R^2$, where

$$A = \frac{8}{1+\kappa^2 - B} \tag{3}$$

and

$$B = \frac{1-\kappa^2}{\ln(1/\kappa)} \tag{4}$$

The latter pressure scale is chosen so that the dimensionless pressure gradient along the domain, in the incompressible flow is equal to unity. Using the above scalings, the dimensionless forms of the equation of state, the continuity equation and z- and r-momentum equations become:

$$\rho = 1 + \varepsilon P \tag{5}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{\partial}{\partial z}(\rho u_z) = 0 \tag{6}$$

$$\alpha Re \rho \left(u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} \right) = -A\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r} \right) + \alpha^2 \frac{\partial^2 u_z}{\partial z^2} + \alpha^2 \left(\chi + \frac{1}{3} \right) \left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r) \right) + \frac{\partial^2 u_z}{\partial z^2} \right] \tag{7}$$

$$\alpha^3 Re \rho \left(u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} \right) = -A\frac{\partial P}{\partial r} + \alpha^2 \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r) \right) + \alpha^4 \frac{\partial^2 u_r}{\partial z^2} + \alpha^2 \left(\chi + \frac{1}{3} \right) \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r) \right) + \frac{\partial^2 u_z}{\partial r \partial z} \right] \tag{8}$$

where

$$\chi \equiv \frac{\eta_1}{\eta} \tag{9}$$

is the bulk-to-shear viscosity ratio,

$$\alpha \equiv \frac{R}{L} \tag{10}$$

is the aspect ratio of the outer cylinder, and Re and ε are, respectively, the Reynolds and compressibility numbers, which are defined by

$$Re \equiv \frac{\rho_0 UR}{\eta} \tag{11}$$

and

$$\varepsilon \equiv \frac{A\eta\beta LU}{R^2} \tag{12}$$

The Mach number takes the form

$$Ma \equiv \sqrt{\frac{\varepsilon\alpha Re}{A\gamma}} \tag{13}$$

where γ being the heat capacity ratio (or adiabatic index). In this work we consider subsonic flows so that $Ma \ll 1$.

The system of partial differential Eqs. (5)–(8) is supplemented by appropriate boundary conditions, which are shown in Fig. 1. Along the two walls, it is assumed that no slip occurs and the transverse velocity component vanishes (impermeable wall):

$$u_z(z,\kappa) = u_z(z,1) = u_r(z,\kappa) = u_r(z,1) = 0, \quad 0 \le z \le 1 \tag{14}$$

At the exit plane, the dimensionless mass flow rate is set at a value of 1:

$$2\int_k^1 \rho u_z r\, dr = 1 \tag{15}$$

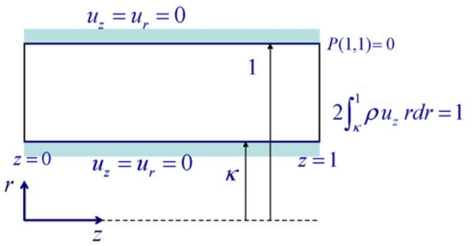

Fig. 1. . Dimensionless geometry and boundary conditions for the compressible annular Poiseuille flow.

Finally, the pressure is set to zero at $z = r = 1$:

$$P(1,1) = 0 \tag{16}$$

As in Venerus [3], no boundary conditions are specified at the inlet plane.

3. Perturbation solution

Since the system of Eqs. (5)–(8) cannot be solved analytically, we seek an approximate perturbation solution. Following Venerus [3], Schwartz [4], and Taliadorou et al. [1], the compressibility number, ε, is used as the perturbation parameter. Due to the complexity of the perturbation solution, this is derived only up to the first order, under the assumption of zero radial velocity. The methodology is described in [1] and only the perturbation solution is given here:

$$\rho = 1 + \varepsilon(1-z) + O(\varepsilon^2) \tag{17}$$

$$P = 1 - z + \varepsilon\left[\frac{\alpha^2}{4}\left(\chi + \frac{1}{3} \right)(1 - r^2 + B\ln r) - \frac{1}{2}(1-z)^2 + \frac{A}{16}\alpha Rec(1-z) \right] + O(\varepsilon^2) \tag{18}$$

$$u_r = O(\varepsilon^2) \tag{19}$$

$$u_z = \frac{A}{4}(1 - r^2 + B\ln r)[1 - \varepsilon(1-z)] + \frac{A^2}{16}\alpha Re\varepsilon\left[K_1 + K_2\ln r - \frac{1}{4}cr^2 + \frac{1}{8}(2 - 4B + 3B^2)r^2 + \frac{1}{16}(B-2)r^4 + \frac{1}{36}r^6 + \frac{1}{2}(B - B^2)r^2\ln r - \frac{1}{8}Br^4\ln r + \frac{1}{4}B^2r^2\ln^2 r \right] + O(\varepsilon^2) \tag{20}$$

where c, K_1, and K_2 are constants defined respectively by

$$c = \frac{A}{192}\left[12(1+\kappa^2+\kappa^4+\kappa^6) - 44(1+\kappa^2+\kappa^4)B + 63(1+\kappa^2)B^2 - 36B^3 \right] \tag{21}$$

$$K_1 = \frac{A}{2304}[-4(2 + 2\kappa^2 - 9\kappa^4 - 9\kappa^6) + (38 - 6\kappa^2 - 132\kappa^4)B - (45 - 81\kappa^2)B^2] \tag{22}$$

and

$$K_2 = \frac{AB}{2304}[-4(2 + 13\kappa^2 + 13\kappa^4 + 2\kappa^6) + (38 + 164\kappa^2 + 38\kappa^4)B - 45(1+\kappa^2)B^2] \tag{23}$$

It is easily verified that the above solution is equivalent to the first-order solution for the axisymmetric Poiseuille flow given in [1] when $\kappa \to 0$. It is also equivalent to the plane Poiseuille flow solution when κ approaches unity.

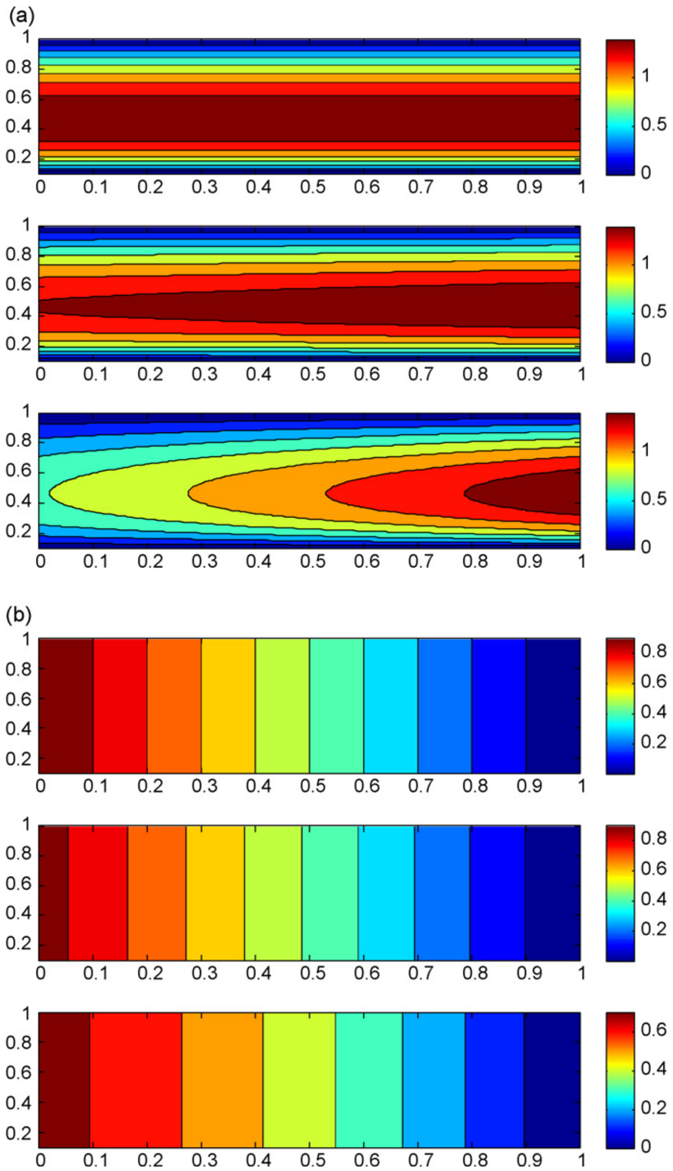

Fig. 2. Effect of ε on (a) the velocity and (b) the pressure contours ($\varepsilon = 0, 0.1, 0.5$); $\kappa = 0.1$, $\alpha = 0.01$, $Re = 0$, $\chi = 0$.

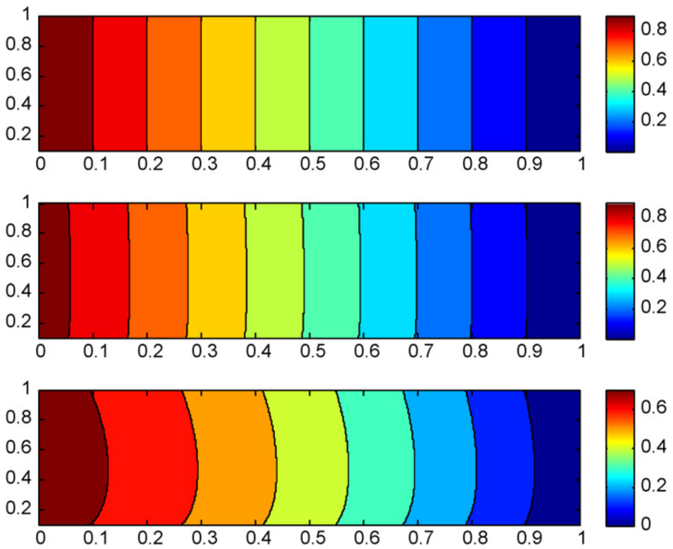

Fig. 3. Effect of ε on the pressure contours ($\varepsilon = 0, 0.1, 0.5$); $\kappa = 0.1$, $\alpha = 1$, $Re = 0$, $\chi = 0$.

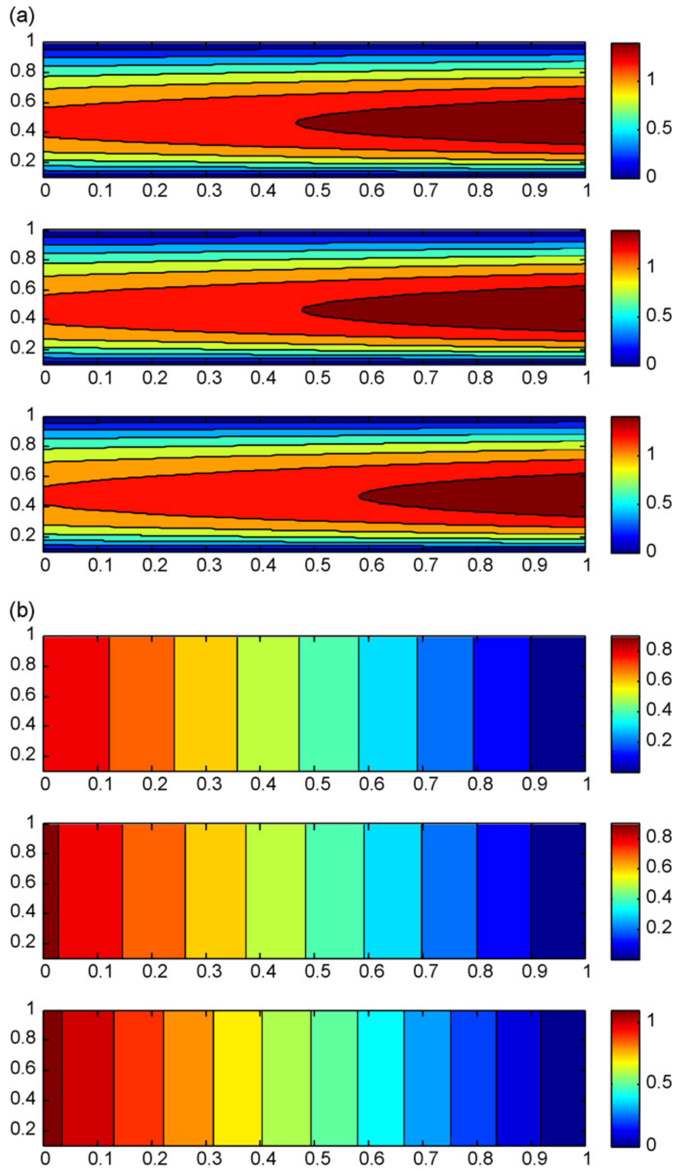

Fig. 4. Effect of Re on (a) the velocity and (b) the pressure contours ($Re = 0, 10, 100$); $\kappa = 0.1$, $\alpha = 0.1$, $\varepsilon = 0.2$, $\chi = 0$.

4. Results

In the present section, the effects of compressibility, the bulk viscosity, the radii ratio, the aspect ratio, and the Reynolds number on the velocity and pressure fields are discussed. The basic features of the velocity and pressure fields described by Eqs. (17)–(20) are the following:

(a) The density is a decreasing function of z, as expected.
(b) The pressure is a function of both z and r and increases with the bulk viscosity, the aspect ratio, and the Reynolds number. The r-dependence at first order in ε becomes stronger as α^2 is increased (i.e. in short channels).
(c) The transverse velocity, u_r, is zero to first order, as in the case of tube and channel flow.
(d) The axial velocity, u_z, deviates from the quasi-parabolic incompressible solution at first order in ε due to fluid inertia. Unlike pressure, this is independent of the bulk viscosity up to the first order. This is also the case in plane and round Poiseuille flows [1].

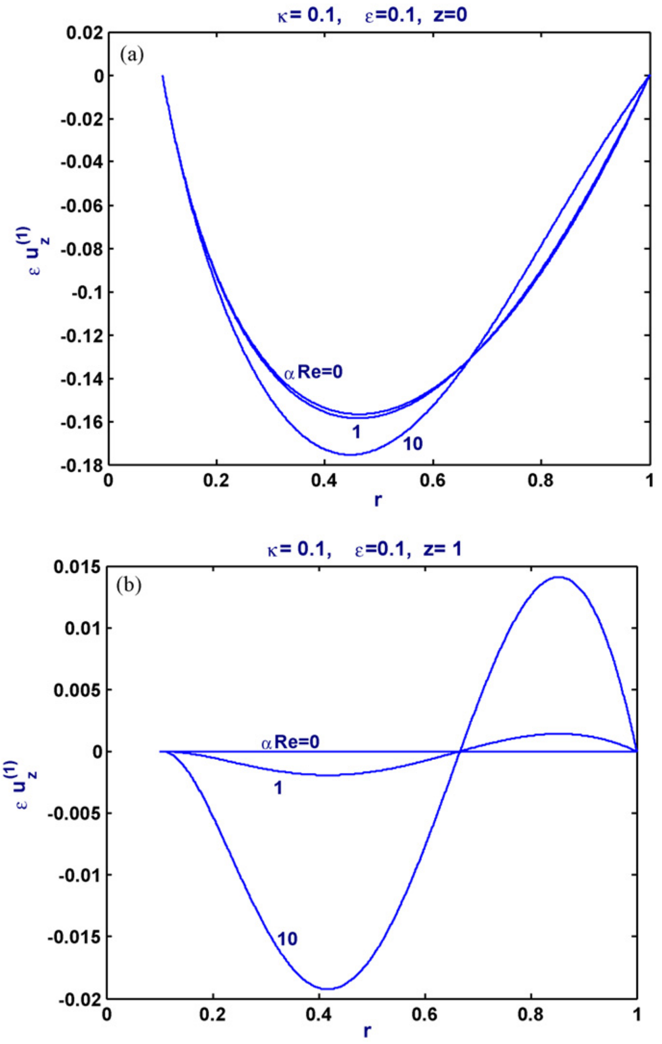

Fig. 5. Deviation of u_z from the incompressible solution at (a) $z = 0$ and (b) $z = 1$; $\kappa = 0.1$ and $\varepsilon = 0.1$.

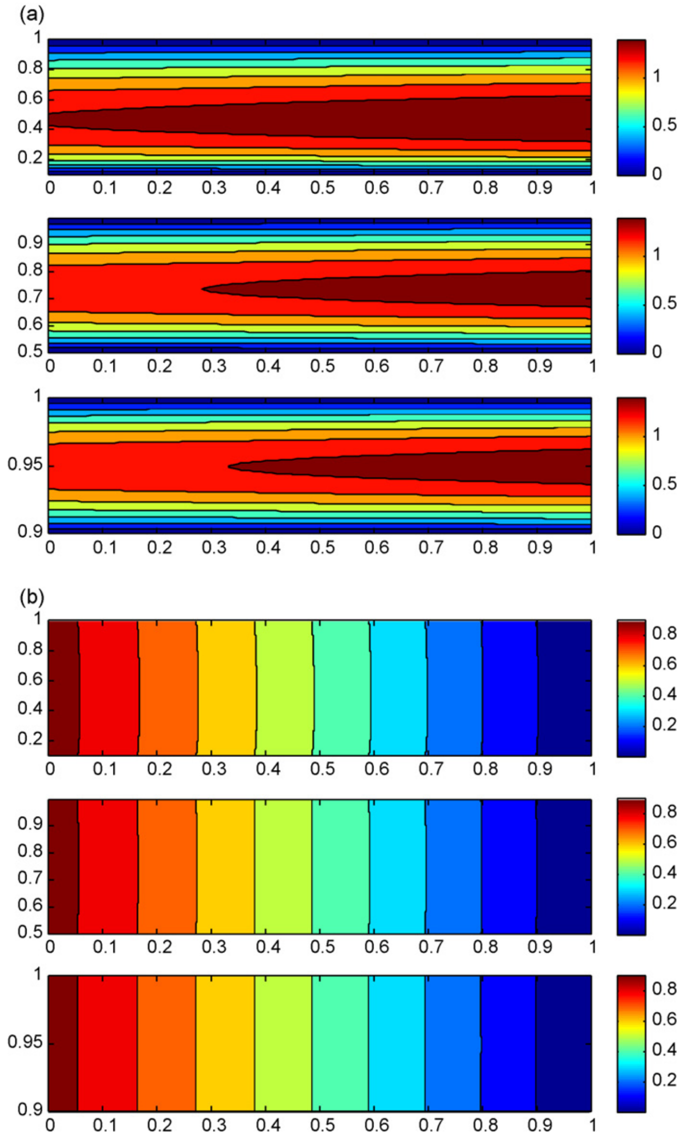

Fig. 6. Effect of κ on (a) the velocity and (b) the pressure contours ($\kappa = 0.1, 0.5, 0.9$); $\alpha = 1$, $\varepsilon = 0.1$, $Re = 0$, $\chi = 0$.

From Eq. (18), it is deduced that the pressure P_W along both walls of the annulus is not dependent on the bulk viscosity:

$$P_W = (1 - z)\left\{1 - \varepsilon\left[\frac{1}{2}(1 - z) - \frac{Ac}{16}\alpha Re\right]\right\} \tag{24}$$

It is also clear that when the Reynolds number is zero, P_W is also independent of the diameter and aspect ratios. The required wall pressure required for driving the flow,

$$\Delta P_W = P_W(0) - P_W(1) = P_W(0) = 1 - \varepsilon\left(\frac{1}{2} - \frac{Ac}{16}\alpha Re\right) \tag{25}$$

is reduced as compressibility is increased, only for small values of αRe. At higher values of αRe, ΔP_W may be an increasing function of ε. It is easily shown that the expression $Ac/16$ is bounded above by 0.2. Since, αRe is of the order of unity, the second term in Eq. (25) may be negligible. Therefore, at first order

$$P_W(0) \simeq 1 - \frac{1}{2}\varepsilon \tag{26}$$

The effect of the term αRe becomes significant only when this is of the order of unity or greater. In the flows of interest, α is 0.01 or less.

Another interesting observation is that for any value of r, the pressure difference between the inlet and the outlet planes is constant (independent of r). It turns out that

$$\Delta P = P_W(0) = 1 - \varepsilon\left(\frac{1}{2} - \frac{Ac}{16}\alpha Re\right) \tag{27}$$

This implies that the pressure profiles for different values of z, differ only by a constant. Let us now consider the pressure profile at the exit plane, which, obviously, is independent of the Reynolds number:

$$P_{exit} = \frac{\alpha^2\varepsilon}{4}\left(\chi + \frac{1}{3}\right)(1 - r^2 + B\ln r) \tag{28}$$

In other words, the pressure varies with r exactly as the incompressible velocity profile. The magnitude of P_{exit} increases linearly with ε and χ and quadratically with α.

The effects of compressibility on the velocity and pressure contours are illustrated in Fig. 2, where results for $\kappa = 0.1$, $\alpha = 0.01$, $Re = 0$, $\chi = 0$, and three different values of ε (0, 0.1 and 0.5), are shown. In the case of incompressible flow the velocity contours are horizontal and those of pressure are vertical. As compressibility is increased, the velocity contours are curved towards the interior of the annular tube. The pressure contours are also curved to the right. This is more clearly seen in shorter tubes, as in Fig. 3 where $\alpha = 1$. It is clear from Eq. (20) that at first order u_z depends on the product αRe. Therefore, the velocity contours are identical to those of Fig. 2 ($\alpha Re = 0$ in both cases).

The effect of the Reynolds number is illustrated in Fig. 4. The velocity contours become more curved as compressibility is increased. The effect of the parameter αRe on the velocity is also illustrated in Fig. 5 where the deviation of the axial velocity from its incompressible counterpart at both the inlet and outlet planes is shown for various values of αRe, $\varepsilon = 0.1$ and $\kappa = 0.1$. Finally, Fig. 6 shows the effect of the diameter ratio on the velocity and pressure contours. Unlike the pressure contours, the velocity contours are more curved at higher values of κ.

5. Conclusions

We have derived, up to the first order in terms of compressibility, the perturbation solution of the annular Poiseuille flow of weakly compressible liquids with constant shear and bulk viscosities, assuming a linear equation of state. The effects of the various parameters involved on the pressure and velocity fields have been discussed. In particular, the pressure difference required to drive the flow in a given tube decreases with compressibility and increases with the Reynolds number. At any axial distance, the pressure distribution is of the form the incompressible velocity profile, i.e. it is minimal at the walls and increases with the compressibility and the bulk viscosity.

An interesting extension of the present work is the consideration of linear slip in compressible Poiseuille flow and the investigation of the possibility to have non-zero transverse velocity to first order due to the variation of the slip velocity with the axial coordinate.

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