Reprinted from AICHE JOURNAL, September 1988

R & D NOTES

Laminar Newtonian Jets at High Reynolds Number and High Surface Tension

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The shapes of Newtonian jets emerging from a long die depend only on two dimensionless parameters, the Reynolds and capillary numbers (*Re* and *Ca*), for negligible gravity and with no axial tension. It is known from experimental data that the extrudate swell for Newtonian creeping flow without surface tension is approximately 1.19 for a plane and 1.13 for an axisymmetric jet (Goren and Wronski, 1966; Nickell et al., 1974). In general, the die-swell ratio decreases as the Reynolds number increases. The jet expands monotonically at low *Re*, contracts monotonically at high Reynolds numbers, and at some intermediate critical range it first contracts and then expands to its final dimension (Goren and Wronski, 1966; Reddy and Tanner, 1978). The limit of the extrudate swell at infinite Reynolds number without surface tension is 0.8660 for the round jet (Harmon, 1955) and 0.8333 for the planar jet (Tillet, 1968).

Surface tension is also important; in general it tends to reduce either expansion or contraction of the jet. For infinite surface tension ($Ca = 0$) neither expansion nor contraction is expected over the entire Reynolds number range. However, at high Reynolds numbers the effect of finite surface tension on the jet profile is less profound than at low Reynolds numbers. The stick-slip problem is equivalent to the plane Newtonian creeping flow with infinite surface tension. Richardson (1970), presented an exact analytic solution to this problem that is often used as a check to the various proposed methods for extrudate-swell computations (Nickell et al., 1974; Ruschak, 1980).

In most cases, the extrudate-swell problem is solved by means of Picard iteration schemes. According to the boundary condition that is used for the relocation of the free surface profile, these iterative procedures are classified as: kinematic iteration, normal-stress iteration, and shear-stress iteration schemes. The kinematic iteration scheme was used by Nickell et al. to solve the creeping axisymmetric Newtonian jet with no surface tension, and by Reddy and Tanner to extend the computations to higher Reynolds numbers and to include the effect of surface tension. The convergence for high *Re*, up to 50, was slow and for $Ca < 1$ convergence could not be achieved. Omodei, (1979, 1980) used the same method to analyze both planar and axisymmetric Newtonian jets at *Re* up to 500 and *Ca* down to 0.2. The convergence at high *Re* was again slow and at $Ca < 0.5$ the computed free-surface profiles were characterized by severe oscillations. Silliman and Scriven (1980) showed that at low surface tension, $Ca > 1$, the kinematic iteration scheme converged very rapidly compared to the normal-stress iteration scheme; however, for high surface tension, $Ca < 1$, the opposite was true. The iterative schemes, based on successive approximations to the free-surface profile, exhibit linear convergence at best (Silliman and Scriven, 1980).

Full Newton iteration is the alternative to the Picard iteration schemes. The free-surface profile is computed simultaneously with the velocities and pressures, and the mesh is updated at each iteration step. The method for nonaxisymmetric two-dimensional flows is described elsewhere (Kistler and Scriven, 1984; Khesghi and Scriven, 1984).

In the present paper, full Newton iteration is used to analyze both axisymmetric and planar jets with surface tension at several Reynolds numbers. In addition, gravity in the direction of flow is included. The method converges quadratically within three to five iterations in wider ranges of Reynolds and capillary numbers than those reported elsewhere.

Formulation

The mathematical formulation of the problem is adequately described elsewhere (Kistler and Scriven, 1984; Kheshgi and Scriven, 1984), and only the final residual equations are presented here. The flow is steady and incompressible and is governed by the momentum and continuity equations along with the boundary conditions of no slip at the wall and no stress at the free surface. The inflow and outflow boundaries are taken at finite distances L_1, L_2, sufficiently far from the exit so that the flow can be considered fully developed at the inlet, and uniform at the

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outflow plane. Thus the boundary conditions at the outlet are

$$T_N = -\frac{1}{Cah_f}, \quad v = 0 \tag{1}$$

where T_N is the total normal stress, v is the radial velocity, $Ca \equiv \mu U/\sigma$, and h_f is the final jet dimension.

Biquadratic, Φ^i, bilinear, Ψ^i, and quadratic, $\Phi^i|_{\eta=1}$, basis functions are used to expand the unknowns u, p, h and weight the momentum, continuity, and kinematic equations, respectively. The final form of the residuals on the computational isoparametric domain are:

$$R_C^i = \int_{-L_1}^{L_2}\int_0^h \left[\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right) r^\alpha + \alpha v\right]\Psi^i |J|\, d\xi\, d\eta, \qquad i = 1, 2, \ldots, M \tag{2}$$

$$\begin{aligned} R_u^i = &-\int_{-L_1}^{L_2}\int_0^h \left[T^{zz}\frac{\partial \Phi^i}{\partial z} + T^{zr}\frac{\partial \Phi^i}{\partial r} + Re\Phi^i\left(u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r}\right) - St\Phi^i\right] r^\alpha |J|\, d\xi\, d\eta \\ &- \frac{1}{Ca}\int_0^{L_2} \frac{h^\alpha z_\xi}{\sqrt{z_\xi^2 + h_\xi^2}}\frac{\partial \Phi^i}{\partial \xi}\bigg|_{\eta=1} d\xi \\ &+ \frac{1}{Ca}\frac{h^\alpha z_\xi}{\sqrt{z_\xi^2 + h_\xi^2}}\bigg|_{\eta=1, z=L_2} \\ &- \frac{\alpha}{Ca}\int_0^{h_f} \frac{r}{h_f}\frac{\partial r}{\partial \eta}\Phi^i|_{\xi=1}\, d\eta, \qquad i = 1, 2, \ldots, N \end{aligned} \tag{3}$$

$$\begin{aligned} R_v^i = &-\int_{-L_1}^{L_2}\int_0^h \left[T^{zr}\frac{\partial \Phi^i}{\partial z} + T^{rr}\frac{\partial \Phi^i}{\partial r} + Re\Phi^i\left(u\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial r}\right) + \alpha\frac{T^{\theta\theta}}{r}\Phi^i\right] r^\alpha |J|\, d\xi\, d\eta \\ &- \frac{1}{Ca}\int_0^{L_2}\left[\alpha\Phi^i\sqrt{z_\xi^2 + h_\xi^2} + \frac{h^\alpha h_\xi}{\sqrt{z_\xi^2 + h_\xi^2}}\frac{\partial \Phi^i}{\partial \xi}\bigg|_{\eta=1}\right] d\xi, \qquad i = 1, 2, \ldots, N \end{aligned} \tag{4}$$

$$R_K^i = \int_0^{L_2} (-uh_\xi + vz_\xi)\Phi^i|_{\eta=1} h^\alpha\, d\xi, \quad i = 1, 2, \ldots, S \tag{5}$$

Here, $|J|$ is the Jacobian of the isoparametric transformation, $St = \rho g R^2/\mu U$ is the Stokes number, and α is a constant utilized to include the planar case in the final equations. For the planar jet r is substituted by y and $\alpha = 0$, whereas $\alpha = 1$ for the axisymmetric jet. The nonlinear system of Eqs. 2, 3, 4, and 5 is solved by Newton iteration employing Hood's (1976) frontal technique.

Results

In this section we present results for both planar and axisymmetric jets at Reynolds numbers from 0 to 2,000, and capillary numbers from 10^5 to 10^{-5}. The results reported by Omodei (1979, 1980) are restricted to $Re \leq 500$ and $Ca > 0.2$. The values of L_1 and L_2 were chosen so that the solution was insensitive to increments to these two lengths. $L_1 = 4.0$ was found to be adequate for all Re and St, whereas L_2 had to be increased as Re and St increased. For zero St, $L_2 = 25$ was adequate for $Re \leq 20$, $L_2 = 100$ for $Re \leq 200$, and $L_2 = 500$ for $Re \leq 2{,}000$; for nonzero St, L_2 had to be increased further. The number of elements ranged from 150 to 222, and the number of unknowns from 1,541 to 2,273.

To check the accuracy of the finite-element predictions we first compared the centerline and free-surface velocities, and the free-surface elevation of the planar jet at $Re = 0$ and $Ca = 10^{-5}$ to the analytic solution for the stick-slip problem (Richardson, 1970). The predictions agreed with theory to within 0.1%. The predictions here were obtained as a limiting case, at $Re = 0$ and $Ca = 10^{-5}$, of the general solution, i.e., the location of the free surface was not fixed *a priori;* the expected planar interface in this limiting case was predicted to within 0.01%.

The computed surface profiles of a round jet, without gravity and surface tension at several Re are shown in Figure 1. The surface tension effects at Reynolds numbers 0, 8, 10, and 100 are shown in Figure 2. Compared to earlier analyses (Omodei, 1980), no oscillations are observed as Ca decreases. The surface tension reduces either expansion or contraction of the jet; at infinitely large surface tension ($Ca \rightarrow 0$), there is no swelling at all, as expected. At moderate Reynolds numbers, from 5 to 15, the surface profile is characterized by a necking. In this critical range, as surface tension increases the die-swell first increases but then starts to decrease, as shown in Figure 2b. Our results are identical to those of Omodei (1979, 1980) for $Ca > 0.5$. At lower capillary numbers his results exhibit oscillations and are limited to $Ca > 0.2$.

As shown in Figure 3, the axisymmetric jet swells less than its planar counterpart at low Reynolds number. At high Reynolds number the opposite is observed. For the planar jet, the die-swell

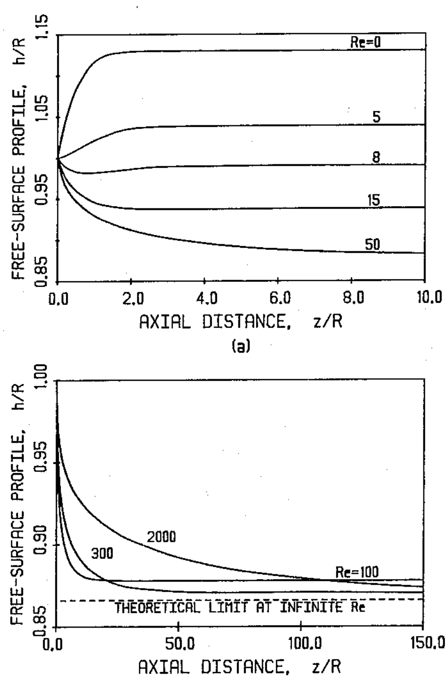

Figure 1. Predicted free-surface profiles of a round jet.
Various $Re = \rho UR/\mu$, zero surface tension

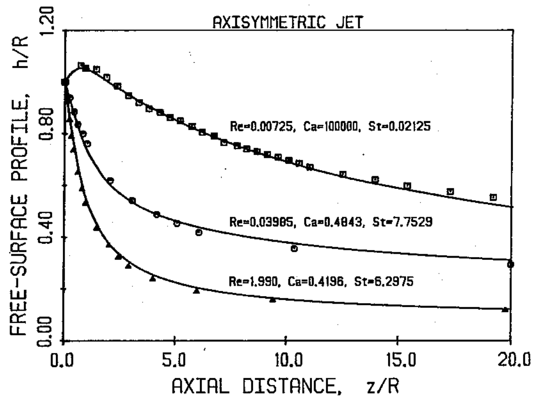

Figure 5. Predicted free-surface profiles of a round jet with gravity.

Comparison with data from Trang and Yeow (1986; top curve), and Adachi and Yoshioka (1984)

verges fast in practically any range of dimensionless numbers examined. Reynolds numbers from zero to transition to turbulence, capillary numbers from zero to infinity, and Stokes numbers from zero to 10. The predictions agree with the analytic solution of the stick-slip problem in the limiting case of zero Reynolds number and infinite surface tension, approach the asymptotic values of the extrudate swelling at infinitely large Reynolds numbers, and compare well with experimental data at several Reynolds, capillary, and Stokes numbers. The advantages of full Newton iteration compared to other methods are:

- Calculations at wider ranges of Reynolds and capillary numbers
- Solution of the gravity-drawn-jet problem on a finite two-dimensional domain
- Fast convergence with the Jacobian for information of linear stability analysis.

Acknowledgment

This research is sponsored by grants-in-aid from General Motors Corp., whose support is gratefully acknowledged.

Notation

Ca = capillary number, $\mu U/\sigma$
g = gravitational acceleration
h = elevation of free surface
h_f = final jet dimension
$|J|$ = Jacobian of isoparametric transformation
L_1, L_2 = distances of inlet and outlet from exit
M, N, S = numbers of pressure, velocity, and free-surface elevation unknowns
p = dimensionless pressure
R = radius
$R_C^i, R_K^i, R_u^i, R_v^i$ = continuity, kinematic, and momentum residuals
Re = Reynolds number, $\rho UR/\mu$
St = Stokes number, $\rho g R^2/\mu U$
$T^{rr}-, T^{rz}-, T^{zz}-, T^{\theta\theta}-$ = rr, rz, zz, and $\theta\theta$ components of stress tensor
T_N = normal stress
u = axial velocity component
U = mean velocity
v = radial velocity component

Greek letters

α = Parameter, Eqs. 2, 3, 4, 5
μ = viscosity
ξ, η = isoparametric coordinates
ρ = density
σ = surface tension
Φ^i, Ψ^i = biquadratic, bilinear basis functions

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Manuscript received Nov. 17, 1987, and revision received Jan. 11, 1988.

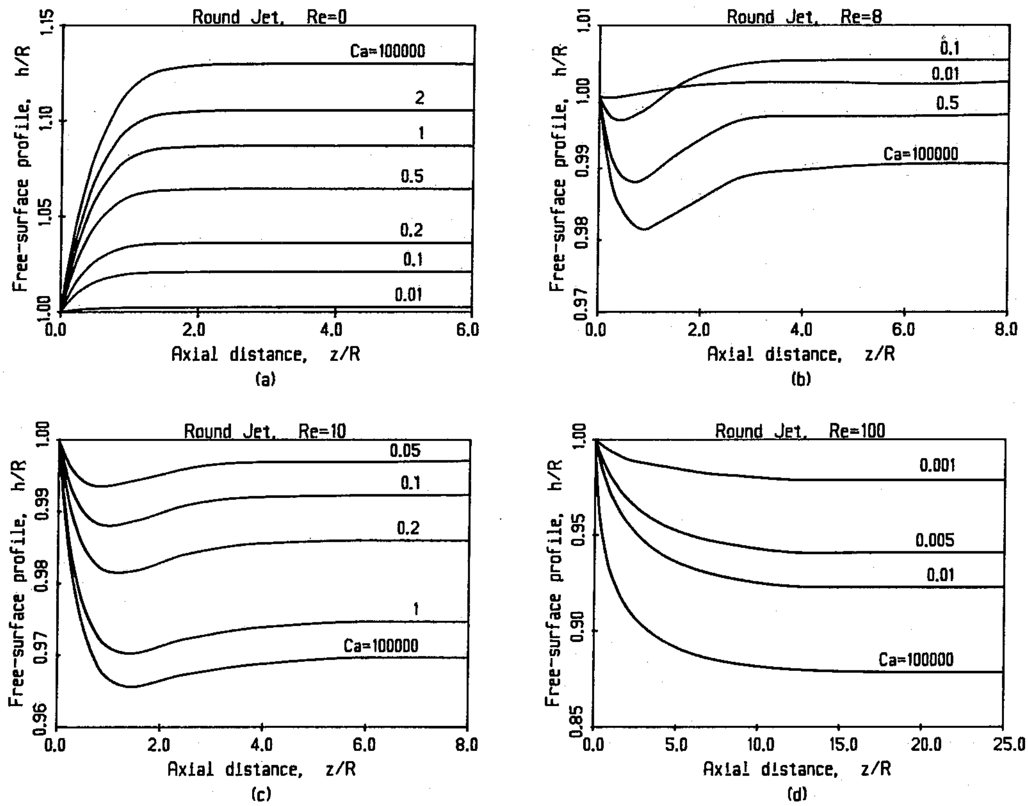

Figure 2. Predicted free-surface profiles of a round jet.

Various $Re = \rho UR/\mu$, $Ca = \mu U/\sigma$

ratio is 1.186 for $Re = 0$, 1 for approximately $Re = 9$, and 0.835 for $Re = 2{,}000$, approaching the theoretical limit 0.8333 at infinite Re (Tillet, 1968); for the axisymmetric jet it is 1.128 for $Re = 0$, 1 for approximately $Re = 7$, and 0.867 for $Re = 2{,}000$, approaching the theoretical limit 0.8660 for infinite Re (Harmon, 1955).

The predictions were tested against the data of Gear et al. (1982) at several Reynolds and capillary numbers. The predicted surface profiles are compared with the data in Figure 4. The computed and experimental final diameters agree to within 1%. An additional feature of this work is that gravity can easily be included. No difficulty was found in solving the gravity-drawn jet on a finite, two-dimensional domain. The boundary condition $v = 0$ at the outflow plane was adequate for nonzero St, inasmuch as the outflow plane was taken sufficiently far from the domain of interest. In Figure 5, we compare our predictions with data taken from Trang and Yeow (1986) and Adachi and Yoshioka (1984). The two agree to within less than 3%.

Conclusions

Finite elements with full Newton iteration have been used to analyze round and planar Newtonian jets. The method con-

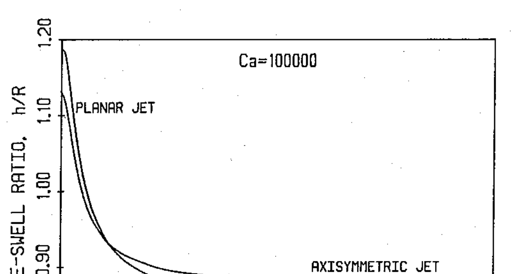

Figure 3. Predicted extrudates-swell ratio vs. Reynolds number at zero surface tension, for both planar and round jets.

----- Asymptotic values at infinite Reynolds number

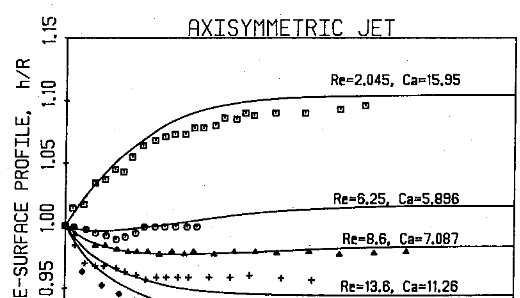

Figure 4. Predicted free-surface profiles of a round jet without gravity.

Comparison with data from Gear et al. (1982)