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A SINGULAR FUNCTION BOUNDARY INTEGRAL METHOD FOR THE LAPLACE EQUATION

GEORGIOS C. GEORGIOU

Department of Mathematics and Statistics, University of Cyprus, Kallipoleos 75, P.O. Box 537, CY–1678 Nicosia, Cyprus

LORRAINE OLSON

Department of Mechanical Engineering, University of Nebraska, Lincoln, NE 68588–0656, U.S.A.

AND

YIORGOS S. SMYRLIS

Department of Mathematics and Statistics, University of Cyprus, Kallipoleos 75, P.O. Box 537, CY–1678 Nicosia, Cyprus

SUMMARY

The authors present a new singular function boundary integral method for the numerical solution of problems with singularities which is based on approximation of the solution by the leading terms of the local asymptotic expansion. The essential boundary conditions are weakly enforced by means of appropriate Lagrange multipliers. The method is applied to a benchmark Laplace-equation problem, the Motz problem, giving extremely accurate estimates for the leading singular coefficients. The method converges exponentially with the number of singular functions and requires a low computational cost. Comparisons are made to the analytical solution and other numerical methods.

KEY WORDS Laplace equation; singularities; boundary integral method

1. INTRODUCTION

Singularities often arise in engineering problems when there is a sudden change in the boundary conditions or the boundary itself. Standard numerical methods like the finite-element, boundary-element and finite-difference methods perform poorly in the neighbourhood of singular points. To achieve satisfactory accuracy and convergence rates, special methods taking into account the form of the local solution are often required. Incorporating the form of the singularity in the numerical scheme is generally more effective than mesh refinement. Special numerical methods for singular problems are reviewed in Reference 1.

The form of the singularity for Laplace-equation or biharmonic-equation problems is easily obtained by a local analysis using separation of variables. For the two-dimensional Laplace equation, the asymptotic solution in polar co-ordinates (r, θ), centred at the singular point, is given by

$$u(r,\theta) = \sum_{j=1}^{\infty} \alpha_j r^{\mu_j} f_j(\theta) \qquad (r,\theta) \in V \tag{1}$$

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where V is a simply connected domain, u is the dependent variable, α_j are the unknown singular coefficients, μ_j are the singularity powers arranged in ascending order, and the functions $f_j(\theta)$ represent the θ dependence of the eigensolution. The values of μ_j and the form of $f_j(\theta)$ are determined by the local analysis. The functions

$$W^j = r^{\mu_j} f_j(\theta) \tag{2}$$

satisfy the governing equation in the domain and the boundary conditions along the parts of the boundary that cause the singularity. The singular coefficients α_j depend on the global problem and are often desirable in many applications. As an example, in fracture mechanics, the first coefficient is the stress intensity factor, a measure of the stress at which fracture occurs.[2]

In a previous paper,[1] we developed the integrated singular basis function method (ISBFM), a finite-element method based on the direct subtraction of the leading terms of the singular local solution from the original mathematical problem. Finite elements are thus used to approximate the 'smooth' part of the solution. Since the basis functions derived from the local solution satisfy the governing equations, a double application of the divergence theorem reduces all integrals involving the singular terms to boundary integrals with non-singular integrands. Lagrange multipliers weakly enforce the originally essential boundary conditions, coupling the ordinary polynomial finite element basis functions with the singular basis functions. The ISBFM has been used for the solution of standard Laplace-equation problems yielding accurate estimates for the leading singular coefficients.[1] It has also been extended to solve singular fluid mechanics problems like the stick-slip and the extrudate-swell problems.[3] Compared to other singular methods, the ISBFM eliminates the need for high-order integration in the neighbourhood of the singularity and improves the overall accuracy. It also accelerates the convergence with regular mesh refinement and converges rapidly with the number of singular functions.

In the present paper, we propose a singular function boundary integral method based on a modification of the ISBFM. The solution is approximated by the leading terms of the singularity expansion:

$$\bar{u} = \sum_{j=1}^{N_\alpha} \alpha_j r^{\mu_j} f_j(\theta) \tag{3}$$

where N_α is the number of basis functions $W^j = r^{\mu_j} f_j(\theta)$. As already pointed out, the basis functions exactly satisfy the governing equation and the boundary conditions along the boundary causing the singularity. With the double application of the divergence theorem all the discretized equations are reduced to boundary integrals. As with the ISBFM, Lagrange multipliers are used to apply the essential boundary conditions. The advantages of this formulation are the following: (a) the dimension of the problem is reduced by one and, consequently, the computational cost is considerably lower; (b) the convergence of the solution with the number of singular functions is exponential.

We demonstrate the method on a Laplace equation problem, the Motz problem.[4] This is considered as a benchmark problem for testing the various singular methods proposed in the literature.[3] Figure 1 shows the geometry, the governing equations and the boundary conditions for the Motz problem as modified by Wait and Mitchell.[5] A singularity arises at $x = y = 0$, where the boundary condition suddenly changes from $u = 0$ to $\partial u/\partial y = 0$. The local solution is given by

$$u = \sum_{j=1}^{\infty} \alpha_j r^{(2j-1)/2} \cos\left[\left(\frac{2j-1}{2}\right)\theta\right] \tag{4}$$

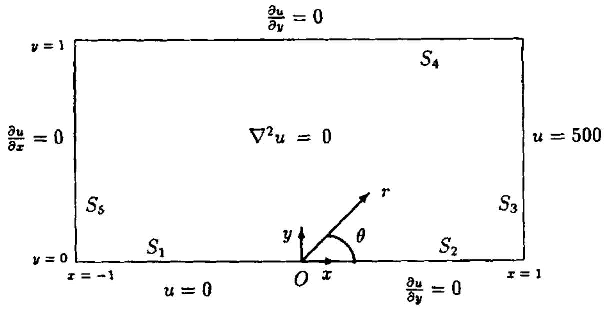

Figure 1. The Motz problem

The above expansion is valid in the entire solution domain V.[6] In fact, its radius of convergence is at least as large as 2.[7] Rosser and Papamichael[7] obtained the exact solution of the Motz problem using a conformal mapping technique. They computed accurate approximations to the first 20 coefficients, expressing them in terms of the coefficients in the series expansions of various elliptic functions and integrals involved in their conformal maps.[8]

Many special numerical schemes have been proposed for the solution of the Motz problem, including finite-difference, global-element, boundary-element, finite-element and other methods. We refer the reader to our previous paper[1] for more details. We should also add here the early works of Morley[9] and Yamamoto[10] who used special finite-element techniques for singular problems and those of Symm[11] and Papamichael and Symm,[12] who developed singular boundary integral methods. In what follows we will make comparisons of our numerical results to the exact values of Rosser and Papamichael[7] and to the two numerical methods that are, to our knowledge, the most accurate in the literature: the ISBFM and the boundary method of Li *et al.*.[13] In the latter method, expansion (4) is used to approximate the solution but the boundary conditions are satisfied in a least-squares sense.

The formulation of the boundary element method is presented in Section 2. Even though we focus on the Motz problem, we should stress that the method is quite general and it can be used for other singular problems. The results are presented in Section 3, where we make comparisons to the exact solution of Rosser and Papamichael,[7,8] the boundary-element results of Li *et al.*,[13] and the values obtained with the ISBFM.[1] The conclusions are summarized in Section 4.

2. THE NUMERICAL METHOD

We present the boundary element formulation for the Motz problem. The same ideas can straightforwardly be extended for other singular elliptic problems. The solution u is approximated using as basis functions the leading terms of the asymptotic expansion in equation (4):

$$\bar{u} = \sum_{j=1}^{N_a} a_j W^j \tag{5}$$

The singular basis functions are used to weight the governing equation in the Galerkin sense. The volume integrals resulting from a double application of Green's theorem are identically zero because the basis functions W^i are harmonic. One then obtains the following discretized integral equations:

$$\int_S \left(\frac{\partial \bar{u}}{\partial n} W^i - \bar{u} \frac{\partial W^i}{\partial n} \right) \mathrm{d}S = 0 \qquad i = 1, 2, \ldots, N_a \tag{6}$$

where S is the boundary of the domain V consisting of five differents parts as shown in Figure 1, and n denotes the co-ordinate normal to the boundary. The integrals along boundaries S_1 and S_2 are zero because the corresponding boundary conditions are identically satisfied by the basis functions W^i. Moreover, the normal derivatives of $\bar{u}$ along S_4 and S_5 are zero. Equation (6) becomes

$$\int_{S_3} \left(\frac{\partial \bar{u}}{\partial x} W^i - \bar{u} \frac{\partial W^i}{\partial x} \right) \mathrm{d}y - \int_{S_4} \bar{u} \frac{\partial W^i}{\partial y} \mathrm{d}x + \int_{S_5} \bar{u} \frac{\partial W^i}{\partial x} \mathrm{d}y = 0, \qquad i = 1, 2, \ldots, N_a \tag{7}$$

As in Reference 1, to impose the essential condition $u = 500$ along S_3, we introduce Lagrange multipliers λ_j. Let λ denote the normal derivative of $\bar{u}$ on S_3. We use quadratic basis functions M^j to expand λ:

$$\lambda = \frac{\partial \bar{u}}{\partial x} = \sum_{j=1}^{N_\lambda} \lambda_j M^j \tag{8}$$

where N_λ is the number of Lagrange multipliers. To define the quadratic basis functions M^j, we divide the boundary S_3 into 3-node boundary elements. The unknowns λ_j are then the nodal values of the normal derivative of $\bar{u}$. The basis functions M^i are used to weight the essential boundary condition along S_3. We thus obtain the following system of equations:

$$\int_{S_3} \left(\lambda W^i - \bar{u} \frac{\partial W^i}{\partial x} \right) \mathrm{d}y - \int_{S_4} \bar{u} \frac{\partial W^i}{\partial y} \mathrm{d}x + \int_{S_5} \bar{u} \frac{\partial W^i}{\partial x} \mathrm{d}y = 0 \qquad i = 1, 2, \ldots, N_a \tag{9}$$

$$\int_{S_3} \bar{u} M^i \, \mathrm{d}y = 500 \int_{S_3} M^i \, \mathrm{d}y \qquad i = 1, 2, \ldots, N_\lambda \tag{10}$$

The discretized equations (9) and (10) constitute a linear system of $N_a + N_\lambda$ equations. Let us now denote the equations of (9) and (10) by **X1** and **X2**, respectively, use the symbols **A** and **Λ** for the vectors of the two sets of unknowns, and also denote by **B** the contributions on the RHS of equation (10). Then the above system of equations is of the form

$$\begin{bmatrix} \dfrac{\partial \mathbf{X1}}{\partial \mathbf{A}} & \dfrac{\partial \mathbf{X1}}{\partial \mathbf{\Lambda}} \\ \dfrac{\partial \mathbf{X2}}{\partial \mathbf{A}} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{\Lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{O} \\ \mathbf{B} \end{bmatrix} \tag{11}$$

As with the ISBFM, the stiffness matrix is symmetric. We observe that we should have $N_a \geqslant N_\lambda$ if the stiffness matrix is to be non-singular (equations in (10) should be fewer than those in (9)).

For the numerical integration, the elements are subdivided into ten sub-elements over which a 15-point Gauss–Legendre quadrature is employed. Different tests with lower-order quadratures and/or more element subdivisions showed that the quadrature used is satisfactory for $N_a < 100$.

3. RESULTS

As mentioned in the previous Section, we should have $N_a \geq N_\lambda$ if the stiffness matrix is to be non-singular. Furthermore, for higher values of N_a, a stronger coupling of equations (9) and (10) is achieved and the condition of the stiffness matrix is improved. The calculated values of the Lagrange multipliers are characterized by oscillations when N_a is not sufficiently high. Hence, a good measure of the quality of the solution is the smoothness of λ. In Figure 2 we plot the values of λ obtained with $N_a = 52$ and 70 ($N_\lambda = 17$). For $N_a = 52$, λ oscillates wildly especially at high y. Increasing N_a results in a stronger coupling of equations (9) and (10) and the wiggles gradually disappear. On the other hand, our calculations show that the convergence of the solution worsens when N_a gets high (greater than 90) and therefore N_λ should be kept as small as possible. As we will see below, this lack of convergence does not affect the values of the leading coefficients; it only affects the 8th decimal (or the 4th significant) digit of the last singular coefficients.

In Table I we show the effect of N_λ on the calculated values of $\alpha_1, \alpha_5, \alpha_{10}, \alpha_{15}$ and α_{20}, obtained with $N_a = 75$. One notices that the values of the singular coefficients converge rapidly with N_λ and that highly accurate estimates are obtained at least for the 20 leading coefficients. This is shown in Figure 3, where we plot the absolute error for α_{15} as a function of $N_\lambda (N_a = 75)$.

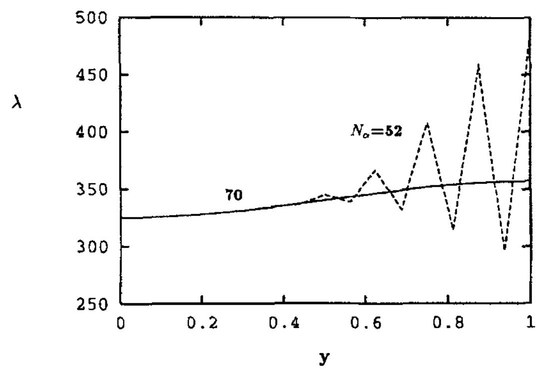

Figure 2. Calculated Lagrange multipliers for $N_a = 52$ and 70 ($N_\lambda = 17$)

Table I. Convergence of the solution with N_λ ($N_a = 75$)

N_λ	α_1	α_5	α_{10}	α_{15}	α_{20}
3	401·1603987635	1·4436773630	0·0411230925	0·0117259851	0·0044337214
5	401·1624253816	1·4400452114	0·0153638444	0·0019658131	0·0013995401
11	401·1624537452	1·4402724379	0·0153842469	0·0002706916	−0·0000119044
17	401·1624537452	1·4402727181	0·0153843735	0·0002715126	−0·0000052961
21	401·1624537452	1·4402727170	0·0153843745	0·0002715122	−0·0000052956
25	401·1624537452	1·4402727170	0·0153843745	0·0002715122	−0·0000052957
29	401·1624537452	1·4402727170	0·0153843745	0·0002715122	−0·0000052958
33	401·1624537452	1·4402727170	0·0153843745	0·0002715122	−0·0000052958
41	401·1624537452	1·4402727170	0·0153843745	0·0002715122	−0·0000052959
Exact	401·1624537452	1·4402727170	0·0153843745	0·0002715122	−0·000005295

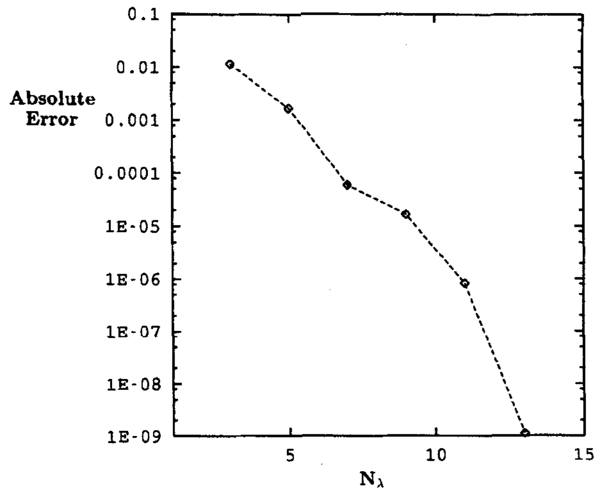

Figure 3. Absolute error of α_{15} as a function of N_λ ($N_a = 75$)

In Table II we show the convergence of the solution with N_a when $N_\lambda = 25$. We would like to make two remarks: (a) for $N_a > 90$ the stability of the solution appears to start deteriorating (the high accuracy of the leading coefficients is conserved but some oscillations appear after the 8th decimal digit of the high-order coefficients). A similar loss of stability is observed with the boundary method of Li *et al.*;[13] (b) the method gives converged results (up to the 10th decimal digit) only for the first 20 coefficients. For the higher-order coefficients (up to α_{30}) we observe oscillations which do not allow the exact determination of the fourth significant digit. This is shown in Table II, where we list the calculated values of α_{25} for different N_a. The exponential convergence of the method is illustrated in Figure 4, where we plot the absolute error for α_{15} as a function of $N_a (N_\lambda = 25)$.

Finally, in Table III we compare the 'best' values of the first 25 coefficients (obtained with $N_a = 75$ and $N_\lambda = 33$) to the exact values of Rosser and Papamichael,[7] the results obtained with the ISBFM,[1] and the most accurate values calculated by Li *et al.*[13] Note that the last one or two final digits of the exact solution might be in error.[7] The present method yields more accurate values of the singular coefficients and requires a smaller computational effort than the other two methods.

Table II. Convergence of the solution with N_a ($N_\lambda = 25$)

N_a	α_1	α_{10}	α_{20}	α_{25}
25	401·1628822971	0·0205739506	−0·0006210054	0·0000131064
30	401·1624533930	0·0151398760	0·0004413487	−0·0000070329
40	401·1624537452	0·0153843722	−0·0000052964	−0·0000003590
50	401·1624537452	0·0153843825	−0·0000056267	0·0000000889
60	401·1624537446	0·0153843737	−0·0000034623	0·0000002348
70	401·1624537452	0·0153843745	−0·0000052957	0·0000001092
80	401·1624537452	0·0153843745	−0·0000052956	0·0000001085
90	401·1624537452	0·0153843745	−0·0000052954	0·0000001068
Exact	401·1624537452	0·0153843745	−0·000005295	

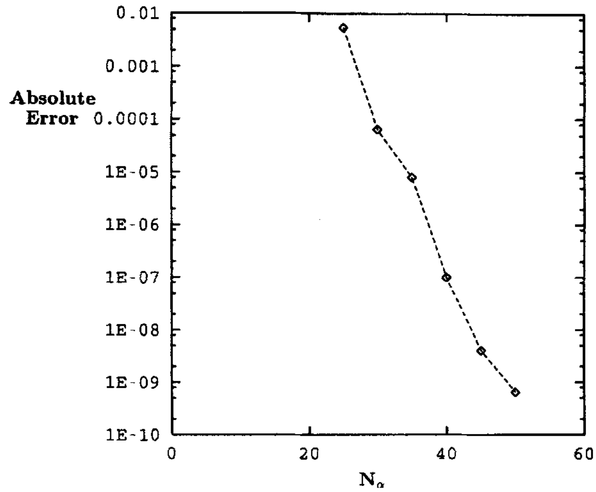

Figure 4. Absolute error of α_{15} as a function of N_a ($N_\lambda = 25$)

Table III. Comparison of the calculated coefficients to those of other methods

i	Exact[7]	ISBFM[1] $N_a = 40$	Li *et al.*[13] $N_a = 35$	Present method $N_a = 75, N_\lambda = 33$
1	401·1624537452	401·1624537452	401·1624537450	401·1624537452
2	87·6559201951	87·6559202595	87·6559201941	87·6559201951
3	17·2379150794	17·2379150363	17·2379150819	17·2379150794
4	−8·0712152597	−8·0712152597	−8·0712152607	−8·0712152597
5	1·4402727170	1·4402727171	1·4402727163	1·4402727170
6	0·3310548859	0·3310548859	0·3310548866	0·3310548859
7	0·2754373445	0·2754373443	0·2754373447	0·2754373445
8	−0·0869329945	−0·0869329946	−0·0869329948	−0·0869329945
9	0·0336048784	0·0336048784	0·0336048781	0·0336048784
10	0·0153843745	0·0153843745	0·0153843747	0·0153843745
11	0·0073023017	0·0073023017	0·0073023019	0·0073023017
12	−0·0031841139	−0·0031841139	−0·0031841138	−0·0031841139
13	0·0012206461	0·0012206461	0·0012206456	0·0012206461
14	0·0005309655	0·0005309655	0·0005309655	0·0005309655
15	0·0002715122	0·0002715122	0·0002715122	0·0002715122
16	−0·0001200463	−0·0001200464	−0·0001200450	−0·0001200464
17	0·0000505400	0·0000505398	0·0000505387	0·0000505398
18	0·000023167	0·0000231668	0·0000231664	0·0000231669
19	0·000011535	0·0000115352	0·0000115349	0·0000115353
20	−0·000005295	−0·0000052957	−0·0000052931	−0·0000052958
21		0·0000022911	0·0000022895	0·0000022911
22		0·0000010632	0·0000010624	0·0000010635
23		0·0000005312	0·0000005307	0·0000005314
24		−0·0000002473	−0·0000002449	−0·0000002474
25		0·0000001097	0·0000001085	0·0000001087

4. CONCLUSIONS

We have developed a singular function boundary integral method based on approximation of the solution by the leading terms of the local asymptotic expansion, and on the use of Lagrange multipliers for the enforcement of the essential boundary conditions. The method has been applied to the Motz problem, the solution of which is described by a single expansion, giving more accurate estimates for the leading singular coefficients than other numerical methods in the literature. It exhibits an exponential convergence with the number of singular functions.

For other singular problems for which more than one expansion is valid over the entire domain, the method can still be applied by subdividing the domain into several subdomains and using different expansions (or methods) in each of them. The extension of the method to other singular problems is currently under investigation.

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