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A fast numerical scheme for the Poiseuille flow in a concentric annulus

Raja R. Huilgol [a], Georgios C. Georgiou [b,*]

[a] College of Science and Engineering, Flinders University, GPO Box 2100, Adelaide SA 5001, Australia
[b] Department of Mathematics and Statistics, University of Cyprus, PO Box 20537, 1678 Nicosia, Cyprus

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ABSTRACT

A fast numerical scheme is proposed to determine the velocity field of an incompressible fluid in a concentric annulus under a constant pressure gradient. The idea behind the scheme is to find the radius R in the annulus where the shear stress becomes zero. In the region from the inner wall at $R=\kappa$ to R, the shear rate is positive, while it is negative from this radius to the outer wall at $r=1$. Integrating the velocity field from the inner wall, where it is zero, one determines its value at R. This acts as the initial value for the integration of the shear rate over the second region, where the velocity must decrease to zero at the outer wall. Choosing a value for R, iterations continue to find its optimal value till the velocity on the outer wall vanishes to within an acceptable error term, which is 10^{-10} here. The numerical method chosen here delivers this result within 5 to 10 iterations for generalised Newtonian and PTT fluids. For viscoplastic fluids, instead of finding the optimal value for R which lies within the plug, one has to find its counterpart r_1. Noting that the shear stress equals the yield stress at r_1 and that the shear rate is positive from κ to r_2 one finds the velocity at this radius. Since the width of the plug r_2-r_1 within the fluid is known and the velocity is constant across the plug, one integrates the velocity field from r_2 to the outer wall at $r=1$ till the velocity approaches zero to within the chosen error term. Once again, the number of iterations to find the velocity field in Bingham and Herschel-Bulkley fluids is small and lies between 5 and 8. Finally, application of the numerical scheme to determine the velocity field in helical flows is also suggested.

1. Introduction

We consider the Poiseuille flow of an incompressible fluid along a concentric annular tube of radii $R_1<R_2$ and denote by κ the radii ratio $R_1/R_2<1$, as illustrated in Fig. 1. The flow occurs under no-slip conditions along the walls of the tube.

In order to explain the idea behind the numerical scheme, we shall begin by assuming that lengths are scaled with respect to the radius R_2, and the stress components by GR_2, where $G>0$ is the constant pressure drop per unit length. This scaling leads to $dp/dz=-1$, where p is the pressure and the flow occurs in the z-direction. It is well known that the equation of motion for the flow is given by

$$\frac{d\tau_{rz}}{dr}+\frac{1}{r}\tau_{rz}=-1 \tag{1}$$

The complete solution of this differential equation is the sum of the homogeneous part, which is b/r; the particular solution is $-r/2$. Thus:

$$\tau_{rz}(r)=\frac{b}{r}-\frac{1}{2}r, \quad \kappa\le r\le 1 \tag{2}$$

Here, the non-dimensional constant b is unknown which makes it difficult to determine the velocity field $u=u(r)$ in the annulus. To overcome this difficulty, we propose a numerical scheme to search for the optimal value of b, based on the fact that $u(\kappa)=0$ and using a shooting method to enforce the condition $u(1)=0$ to within an acceptable error term ε. The chosen algorithm delivers a value of $\varepsilon=10^{-10}$ in a very small number of iterations, between 5 and 10.

First of all, we demonstrate that $b>0$ as follows. On both of the bounding surfaces, the shear stress $\tau_{rz}<0$, for it opposes the motion. Let $\mathbf{e}_r$ be the unit vector in the radial direction. On the inner surface at $r=\kappa$ the unit external normal $\mathbf{n}$ points towards the axis, or $\mathbf{n}=-\mathbf{e}_r$. On $r=1$, the unit external normal points outwards and thus, $\mathbf{n}=\mathbf{e}_r$. Let $\mathbf{T}$ be the total stress tensor in the fluid. Cauchy's stress principle says that the stress vector $\mathbf{t}$ on the bounding surface is given by $\mathbf{t}=\mathbf{Tn}$. Hence,

$$\tau_{rz}(\kappa)=\frac{b}{k}-\frac{1}{2}\kappa>0, \quad \tau_{rz}(1)=b-\frac{1}{2}<0 \tag{3}$$

It is obvious that these inequalities can hold if and only if $\kappa^2/2<b<1/2$.

* Corresponding author.
E-mail address: georgios@ucy.ac.cy (G.C. Georgiou).

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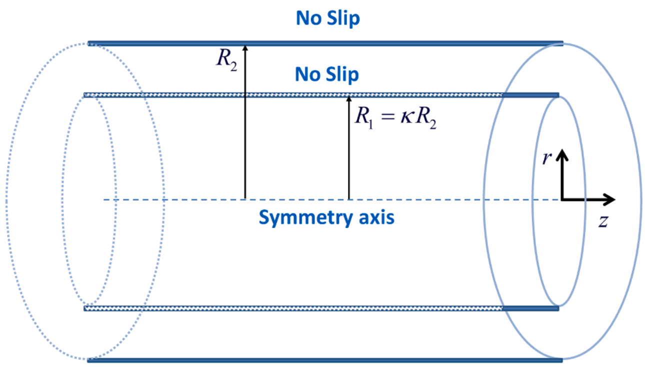

Fig. 1. Geometry and boundary conditions for the annular Poiseuille flow.

The fact that the shear stress τ_{rz} changes its sign from being positive at $r = \kappa$ to being negative at $r = 1$ means that $\tau_{rz} = 0$ somewhere in between. This is true in all fluids, including viscoplastic fluids. Let us denote this radial distance as R, that is $\tau_{rz}(R) = 0$. Obviously, a relationship between b and R exists and this is given by

$$b = R^2/2 \tag{4}$$

The idea of substituting b by $R^2/2$ can be found in Bird, Stewart and Lightfoot [1]. Another way to understand the significance of R is to note that the velocity field $u = u(r)$ attains its maximum at $r = R$.

We shall now offer a brief explanation to show that it is easier to optimise R rather than b. To be specific, in generalised non-Newtonian fluids, including power-law fluids as well as in viscoelastic fluids, such as Phan-Thien - Tanner (PTT) fluids, the shear rate $du/dr > 0$ in (κ, R). If one can find $u = u(R) = u_R$ through numerical integration over this interval using $u(\kappa) = 0$, one can then use u_R as the initial value to determine $u(1)$ through integration in $[R, 1]$. So, one begins by choosing a value for R, calculating the corresponding $u(1)$ and optimising the value of R through the algorithm described below to achieve $u(1) = 0$ to within an acceptable error term.

In viscoplastic fluids, a change to the above procedure has to be made because the unyielded zone lies in $[r_1, r_2]$ where $\kappa < r_1 < r_2 < 1$; moreover, $r_1 < R = \sqrt{2b} < r_2$. Since $du/dr > 0$ in (κ, r_1), we find $u(r_1)$ through numerical integration. On noting that the fluid moves as a plug in $[r_1, r_2]$, it follows that $u(r_1) = u(r_2)$. Thus, one can find $u(1)$ through numerical integration in $[r_2, 1]$. The previous algorithm can be applied, in principle, to find the best value of r_2. However, it is preferable to optimise r_1, since $r_2 - r_1 = 2Bn$, where Bn is the Bingham number and is also the dimensionless yield stress.

It should be noted that the idea of calculating R numerically in the case of generalised Newtonian fluids, or the radii r_1 and r_2 in the case of yield stress fluids has been exploited in the early works of Frederickson and Bird [2] and Hanks [3]. David et al. [4] proposed an explicit empirical formula yielding R for power-law fluids as a function of κ and the power-law exponent n. Much later, Pinho and Oliveira [5] derived the analytical solution for the linear PTT fluid.

In Section 2, we list the constitutive equation for Herschel-Bulkley fluids [6] first, for it is possible to obtain those for the Newtonian, power-law and the Bingham fluids from this model by an appropriate choice of the parameters. Next, we describe the tensorial form of the PTT model [7] and its linear and exponential approximations [7,8]. In Section 3, we explain the procedure for the non-dimensionalization used in this paper, in detail. In Section 4 the algorithm for generalised Newtonian fluids is developed and applied to the annular flow for several flow parameters. Next, in Section 5, a modified version of this algorithm is used to solve the annular flow problems in Herschel-Bulkley and Bingham fluids. In Section 6, we study the flows of both linear and exponential PTT fluids [7,8]. Finally, we offer some comments on the application of our algorithm to helical flows.

2. Constitutive equations

Let $\mathbf{D}$ be the symmetric part of the velocity-gradient tensor defined through

$$\mathbf{D} \equiv \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right] \tag{5}$$

where $\mathbf{u}$ is the velocity vector and the superscript T denotes the transpose. The constitutive equation for a Herschel-Bulkley [6] fluid can be written as follows

$$\begin{cases} \mathbf{D} = \mathbf{0}, & \tau \le \tau_y \\ \boldsymbol{\tau} = 2\left(\dfrac{\tau_y}{\dot{\gamma}} + k\dot{\gamma}^{n-1}\right)\mathbf{D}, & \tau > \tau_y \end{cases} \tag{6}$$

Here, $\boldsymbol{\tau}$ is the viscous stress tensor, $\dot{\gamma} \equiv \sqrt{2tr\mathbf{D}^2}$ and $\tau \equiv \sqrt{tr\boldsymbol{\tau}^2/2}$ are the magnitudes of $2\mathbf{D}$ and $\boldsymbol{\tau}$, respectively. Note that $tr\mathbf{M}$ stands for the trace of any matrix $\mathbf{M}$. The Herschel-Bulkley fluid involves three parameters: the yield stress τ_y below which the fluid is at rest or moves as a rigid plug; the consistency index k, and the power-law exponent n, such that $0 < n \le 1$. Note that $n > 1$ cannot occur, for the zero shear rate viscosity will be zero.

The Eq. (6) defines a Bingham fluid when $n = 1$. In this case, the consistency index k is the constant plastic viscosity, η, of the Bingham fluid. When $\tau_y = 0$, Eq. $(6)_1$ is irrelevant and the power-law model is obtained from Eq. $(6)_2$ with the Newtonian fluid as a special case when $n = 1$ and k is, once again, the constant viscosity, $n > 1$, of this fluid.

The tensorial form of the PTT constitutive equation is given by [7]:

$$f(tr\boldsymbol{\tau})\,\boldsymbol{\tau} + \lambda\overset{\nabla}{\boldsymbol{\tau}} = 2\eta\mathbf{D} \tag{7}$$

where λ is the relaxation time, η is the constant shear viscosity, and $\overset{\nabla}{\boldsymbol{\tau}}$ denotes Oldroyd's upper-convected derivative:

$$\overset{\nabla}{\boldsymbol{\tau}} = \frac{d\boldsymbol{\tau}}{dt} - \boldsymbol{\tau}\cdot\nabla\mathbf{u} - (\nabla\mathbf{u})^T\cdot\boldsymbol{\tau} \tag{8}$$

Here, $d\boldsymbol{\tau}/dt$ is the material derivative of $\boldsymbol{\tau}$, and the dot denotes the trace of the product of the two matrices. The model is reduced to the upper-convected Maxwell model when $f(tr\boldsymbol{\tau}) = 1$.

The most common forms of the PTT model are the linear one [7]:

$$f(tr\boldsymbol{\tau}) = 1 + \frac{\varepsilon\lambda}{\eta}tr\boldsymbol{\tau} \tag{9}$$

and the exponential one [8]:

$$f(tr\boldsymbol{\tau}) = \exp\left(\frac{\varepsilon\lambda}{\eta}tr\boldsymbol{\tau}\right) \tag{10}$$

where ε is a parameter related to the elongational behaviour of the fluid.

3. Non-Dimensionalisation

As mentioned earlier in the Introduction, all lengths are scaled by the radius R_2 of the outer tube, and stress components by GR_2, where $G > 0$ is the constant pressure drop per unit length. The velocity is scaled through $U = G^{1/n}R_2^{1+1/n}/k^{1/n}$. In the case of Newtonian and PTT fluids, the velocity scale becomes $U = GR_2^2/\eta$.

In generalised Newtonian fluids, the only non-zero component of the stress tensor is the shear stress:

$$\tau_{rz} = \eta(\dot{\gamma})\frac{du}{dr} \tag{11}$$

where $\eta(\dot{\gamma}) = \dot{\gamma}^{n-1}$ for power-law fluids, with $\eta(\dot{\gamma}) = 1$ for Newtonian fluids. In Herschel-Bulkley fluids [6], Eq. (11) holds where the material has yielded, i.e., when $\tau > Bn$, where Bn is the Bingham number:

$$Bn = \frac{\tau_y}{GR_2} \tag{12}$$

and the viscosity is given by

$$\eta(\dot{\gamma}) = \frac{Bn}{\dot{\gamma}} + \dot{\gamma}^{n-1} \tag{13}$$

In the unyielded region, $\tau \leq Bn$, and the velocity is constant. In principle, one can combine Eqs. (2), (11) and (13) to obtain du/dr as a function of the radial distance r and integrate to determine the velocity field satisfying the no-slip conditions along the walls of the annular tube.

In the PTT model, there are two non-zero stress components, viz., τ_{rz} and τ_{zz}. In the linear form [7], these are given by

$$\left(1 + 2\varepsilon De^2 \tau_{rz}^2\right)\tau_{rz} = \frac{du}{dr} \tag{14}$$

and

$$\tau_{zz} = 2De\, \tau_{rz}^2 \tag{15}$$

where

$$De = \frac{\lambda G R_2}{\eta} \tag{16}$$

is the Deborah number.

In the exponential model [8], Eq. (14) is replaced by

$$\exp\left(2\varepsilon De^2 \tau_{rz}^2\right)\tau_{rz} = \frac{du}{dr} \tag{17}$$

Combining Eq. (2) with (14) or (17), one can determine the velocity field through numerical integration and in certain cases analytically. We shall now describe the numerical algorithm for generalised Newtonian fluids.

4. Algorithm for generalised Newtonian fluids

For generalised Newtonian fluids with zero yield stress and a viscosity $\eta = \eta(\dot{\gamma})$, where $\dot{\gamma}$ is the shear rate, the adopted algorithm is as follows:

1 Choose a value of R such that $\kappa < R < 1$. This is equivalent to setting $b = R^2/2$.

2 Determine the inner velocity branch in $[\kappa, R]$ by integrating

$$\eta(\dot{\gamma}) \frac{du}{dr} = \frac{b}{r} - \frac{1}{2} r \tag{18}$$

noting that $\dot{\gamma} = du/dr$ and that $u(\kappa) = 0$. In the general case, this has to be done numerically. However, this can also be done analytically for certain values of n when using the power-law model. We let $u_R = u(R)$, the velocity at $r = R$.

3 Calculate the outer velocity branch in $[R, 1]$ through the integration of Eq. (18) noting that $\dot{\gamma} = -du/dr$ and $u_R = u(R)$. Denote by u^* the computed velocity at $r = 1$, i.e., $u^* = u(1)$.

4 Check for convergence and improve the estimate of R. Obviously, if $|u^*|$ is smaller than a tolerance ε the method has converged. Otherwise, there are two possibilities. Let us denote by $R^{(1)}$ the first chosen value for R.

- If $u^* < 0$, the gap $1 - R^{(1)}$ is too large and must be decreased. First, find r^* where $u(r^*) = 0$. Set

$$R^{(2)} = R^{(1)} + \frac{1 - r^*}{2} \tag{19}$$

- If $u^* > 0$, the gap $1 - R^{(1)}$ is too small and must be increased. Here, we find r^* such that $u(r^*) = 0$. Set

$$R^{(2)} = R^{(1)} - \frac{r^* - \kappa}{2} \tag{20}$$

5 Let $j \geq 2$ stand for the subsequent iterations. We now use linear interpolation based on the last two values of R and u^*:

$$R^{(j+1)} = R^{(j)} - \frac{R^{(j)} - R^{(j-1)}}{u^{*(j)} - u^{*(j-1)}} u^{*(j)} \tag{21}$$

In all of the numerical results presented in this work, integration was carried out by uniformly subdividing the flow domain into 100 subintervals over which we applied a 21-point Gauss-Legendre quadrature. A 'normalised' tolerance of 10^{-10} has been employed. More specifically, the tolerance used is $\varepsilon = 10^{-10}\max\{u_{\max}, 10^{-4}\}$, where $u_{\max}$ is the maximum velocity. In each case, the method converges in 5 - 10 iterations, which is quite fast. We shall begin with the results obtained by the use of the aforementioned algorithm for Newtonian fluids.

4.1. Newtonian fluids

In this case $\eta(\dot{\gamma}) = 1$. It is known that the theoretical value for of b is given by [9]:

$$b = \frac{1 - \kappa^2}{4\ln(1/\kappa)} \tag{22}$$

Hence,

$$R = (2b)^{1/2} = \left[\frac{1 - \kappa^2}{2\ln(1/\kappa)}\right]^{1/2} \tag{23}$$

It turns out that for the chosen tolerance of 10^{-10}, the method converges in 5 - 7 iterations, as shown in Table 1, where results for $\kappa = 0.1$, 0.5 and 0.9 are tabulated. A highly accurate estimate for the value of R is obtained where the velocity is a maximum, or the shear stress is zero.

4.2. Power-Law fluids

In these fluids, $\eta(\dot{\gamma}) = \dot{\gamma}^{n-1}$, where $0 < n \leq 1$ is the power-law exponent. The radial distance where the shear stress is zero, i.e., the value of R, can be found theoretically by solving the following integral equation [10]:

$$\int_\kappa^R \left(\frac{R^2}{\xi} - \xi\right)^{1/n} d\xi - \int_R^1 \left(\xi - \frac{R^2}{\xi}\right)^{1/n} d\xi = 0 \tag{24}$$

Essentially, this equation states that the velocity $u(R)$, obtained by integrating du/dr over $[\kappa, R]$, is the same as that over $[R, 1]$.

In Fig. 2 we show the convergence of the results obtained for $n = 1/3$ and $\kappa = 0.1$, 0.5 and 0.9. Again, the convergence is quite fast and highly accurate estimates of R are obtained - see Table 2.

5. Algorithm for viscoplastic fluids

In viscoplastic fluids, the flow will commence when the applied pressure drop overcomes the yield stress at the walls. To find this minimum, we note that the shear stress τ_{rz} on the inner surface at $r = \kappa$ is

Table 1
Convergence of the method for a Newtonian fluid.

κ	Iterations	Calculated R	Theoretical R
0.1	7	0.46365479458549	0.463654794585486
0.5	5	0.73553425503737	0.735534255037358
0.9	5	0.94956097760904	0.949560977609042

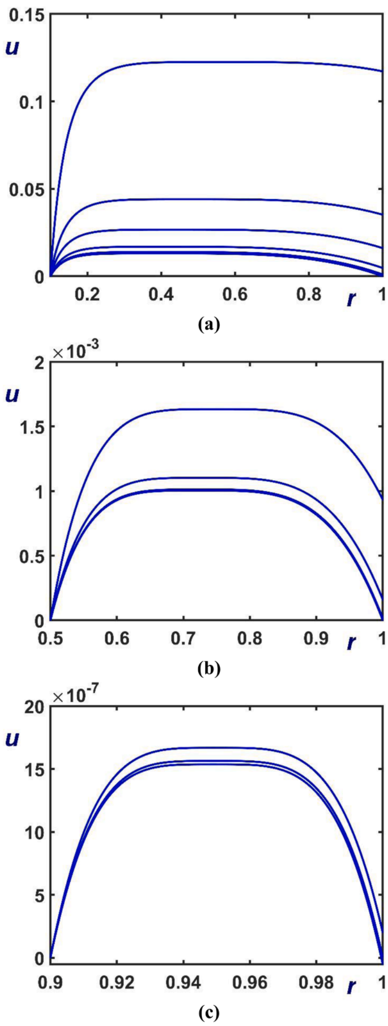

Fig. 2. Convergence of the method for a power-law fluid with $n = 1/3$: (a) $\kappa = 0.1$; (b) $\kappa = 0.5$; (c) $\kappa = 0.9$.

Table 2
Convergence of the method for a power-law fluid with $n = 1/3$.

κ	Iterations	Calculated R	Theoretical R
0.1	9	0.39432072748582	0.39432072748564
0.5	6	0.72396604681510	0.72396604681350
0.9	5	0.94920976384645	0.94920976384653

positive and equals the Bingham number Bn, and that on the outer surface at $r = 1$ is negative and equals $(-Bn)$. Thus, we find the minimum pressure drop per unit length, $G_{\min}$, through

$$Bn = \frac{b}{\kappa}b - \frac{1}{2}G_{\min}\kappa \tag{25}$$

and

$$-Bn = b - \frac{1}{2}G_{\min} \tag{26}$$

which lead to:

$$G_{\min} = \frac{2Bn}{1-\kappa} \tag{27}$$

Hence, when one models the steady flow of a viscoplastic fluid in the annulus, it is assumed that $1 - G_{\min} > 0$.

To begin, we consider Herschel-Bulkley fluids [6] with the following viscosity:

$$\eta(\dot{\gamma}) = \frac{Bn}{\dot{\gamma}} + \dot{\gamma}^{n-1} \tag{28}$$

We note that in the flow domain, there will be an unyielded region moving as a plug. This plug is bounded below at $r = r_1$ and above by $r = r_2$, such that $\kappa < r_1 < r_2 < 1$. The shear stress is Bn at $r = r_1$ and $(-Bn)$ at $r = r_2$. Thus,

$$\tau_{rz}(r_1) = Bn = \frac{b}{r_1} - \frac{1}{2}r_1 \tag{29}$$

$$\tau_{rz}(r_2) = -Bn = \frac{b}{r_2} - \frac{1}{2}r_2 \tag{30}$$

Eq. (29) leads to

$$b = \left(\frac{1}{2}r_1 + Bn\right)r_1 \tag{31}$$

Inserting this value for b into Eq. (30), we find that

$$r_2 = r_1 + 2Bn \tag{32}$$

Hence, the width of the plug is given by $r_2 - r_1 = 2Bn$.

Since $r_2 < 1$, the initial estimate for r_1 must be such that

$$\kappa < r_1 < 1 - 2Bn \tag{33}$$

Thus, in the lower branch, we have to solve

$$\frac{du}{dr} = \left(\frac{b}{r} - \frac{1}{2}r - Bn\right)^{1/n}, \quad \kappa < r < r_1 \tag{34}$$

with $u(\kappa) = 0$. Here, at the radius r_1, we compute $u(r_1)$ and this is the plug velocity in $r_1 \le r \le r_2$. In the upper branch, we have to solve

$$\frac{du}{dr} = -\left(-\frac{b}{r} + \frac{1}{2}r - Bn\right)^{1/n}, \quad r_2 < r < 1 \tag{35}$$

with $u(r_2) = u(r_1)$.

The numerical scheme is based on iterations to adjust the value of r_1, leading to newer values of r_2 through Eq. (32) to ensure that $u(1) = 0$. Thus, we find the location of the plug as well as the velocity field in the flow of a Herschel-Bulkley fluid when n and Bn are chosen.

In the case of the Bingham fluid, with $n = 1$, the radius r_2 can be determined through the solution of the following equation [11]:

$$\left[2\beta\ln(\beta/\kappa) + \beta^2 - 1\right]r_2^2 + 2(1-\beta)(1+\kappa)r_2 - 1 + \kappa^2 = 0 \tag{36}$$

where

$$\beta \equiv \frac{r_1}{r_2} = 1 - \frac{2Bn}{r_2} \tag{37}$$

In Fig. 3 we see the convergence of the results for a Bingham plastic ($n = 1$) with κ=0.1, 0.5 and 0.9 and $Bn = 0.2$, 0.08 and 0.01, respectively. The computed values of r_1 are tabulated in Table 3. These compare well with the solutions obtained from solving the pair of Eqs. (32) and (36).

Fig. 4 shows similar results obtained for a Herschel-Bulkley fluid [6] with $n = 1/2$ and three different values for κ and Bn. It should be noted that the method converges quite fast even though the initial estimate of r_1 results in a velocity which is negative near the wall and sometimes in most of the gap.

6. PTT fluids

In the case of a linear PTT fluid [7], from Eqs. (2) and (14), we have

$$\frac{du}{dr} = \left(1 + 2\varepsilon De^2\tau_{zr}^2\right)\tau_{zr} = \left[1 + 2\varepsilon De^2\left(\frac{b}{r} - \frac{r}{2}\right)^2\right]\left(\frac{b}{r} - \frac{r}{2}\right) \tag{38}$$

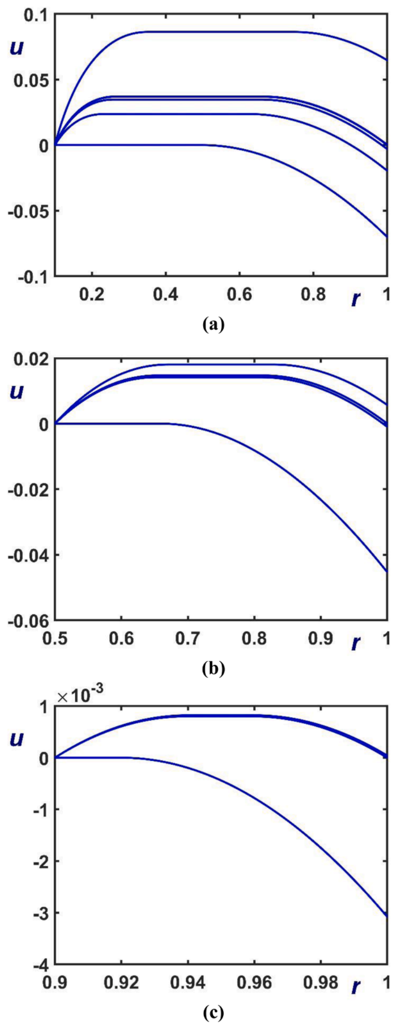

Fig. 3. Convergence of the method for a Bingham plastic fluid ($n = 1$): (a) $\kappa = 0.1,\ Bn = 0.2$; (b) $\kappa = 0.5,\ Bn = 0.08$; (c) $\kappa = 0.9,\ Bn = 0.01$.

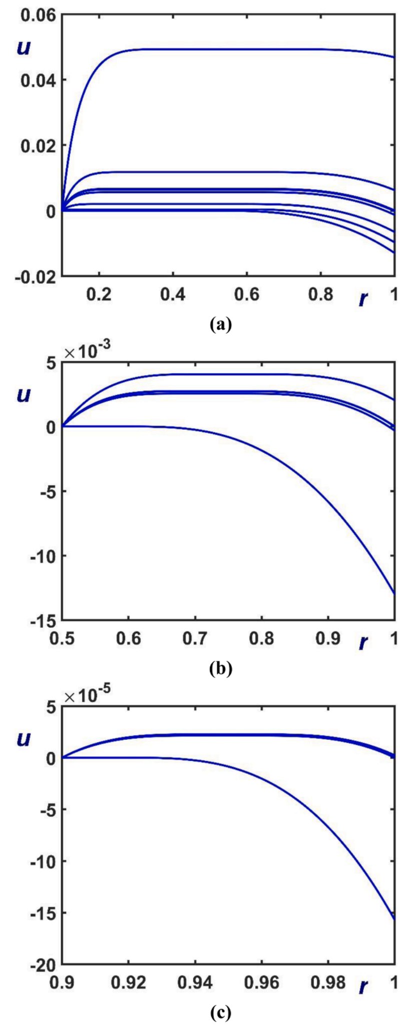

Fig. 4. Convergence of the method for a Herschel-Bulkley fluid with $n = 1/2$: (a) $\kappa = 0.1,\ Bn = 0.2$; (b) $\kappa = 0.5,\ Bn = 0.05$; (c) $\kappa = 0.9,\ Bn = 0.01$.

Table 3
Convergence of the method for a Bingham fluid ($n = 1$).

κ	Bn	Iterations	Calculated r_{01}	Theoretical r_{01}
0.1	0.2	8	0.26294182067723	0.26294182067722
0.5	0.08	7	0.65392337800725	0.65392337800725
0.9	0.01	5	0.93950832427103	0.93950832427100

The numerical procedure adopted by us is the same as that employed for generalised Newtonian fluids in Section 4. The radius R, where the shear stress is zero, can also be found by noting that $b = R^2/2$ and using the fact that $u(\kappa) = u(1) = 0$. Thus,

$$\int_k^1 \left[1+\frac{1}{2}\varepsilon De^2\left(\frac{R^2}{\xi}-\xi\right)^2\right]\left(\frac{R^2}{\xi}-\xi\right)d\xi = 0 \tag{39}$$

This leads to the following non-linear equation for R:

$$\ln(1/\kappa)R^2 - \frac{1}{2}\left(1-\kappa^2\right) + \frac{\varepsilon De^2}{2}\left[\frac{1-\kappa^2}{2\kappa^2}R^6 - 3\ln(1/\kappa)R^4 + \frac{3}{2}\left(1-\kappa^2\right)R^2 - \frac{1}{4}\left(1-\kappa^4\right)\right] = 0 \tag{40}$$

which is equivalent to that solved by Pinho and Oliveira [5].

Representative results showing the convergence of the method observed with $\varepsilon = 0.1$ and various Deborah numbers are provided in Figs. 5 and 6 for $\kappa = 0.1$ and 0.5, respectively. The calculated values of R are compared with the corresponding solutions of Eq. (40) in Tables 4 and 5.

Similar results have been obtained for the exponential PTT fluid [8] in which case

$$\frac{du}{dr} = \left(\frac{b}{r}-\frac{r}{2}\right)\exp\left[2\varepsilon De^2\left(\frac{b}{r}-\frac{r}{2}\right)^2\right] \tag{41}$$

In Fig. 7, the velocity profiles computed with $\kappa = \varepsilon = 0.1$ and $De = 1$, 5 and 10 are shown. Convergence is achieved within 5–10 iterations.

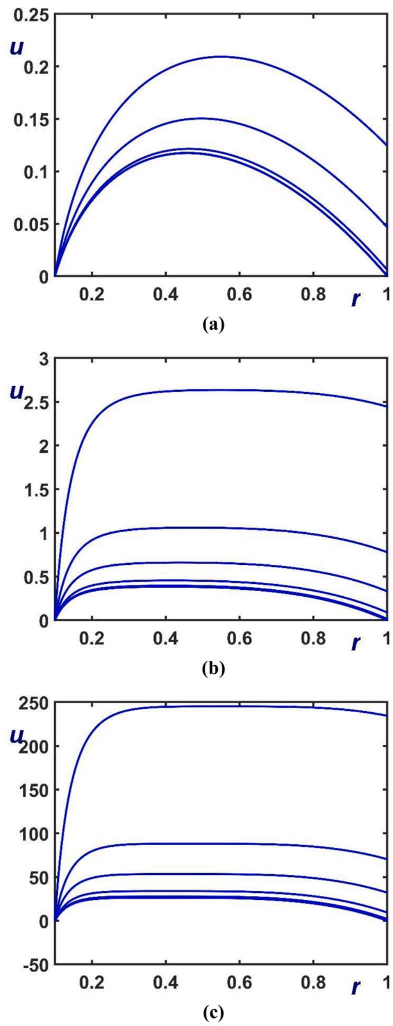

Fig. 5. Convergence of the method for a linear PTT fluid with $\kappa = 0.1$ and $\varepsilon = 0.1$: (a) $De = 1$; (b) $De = 10$; (c) $De = 100$.

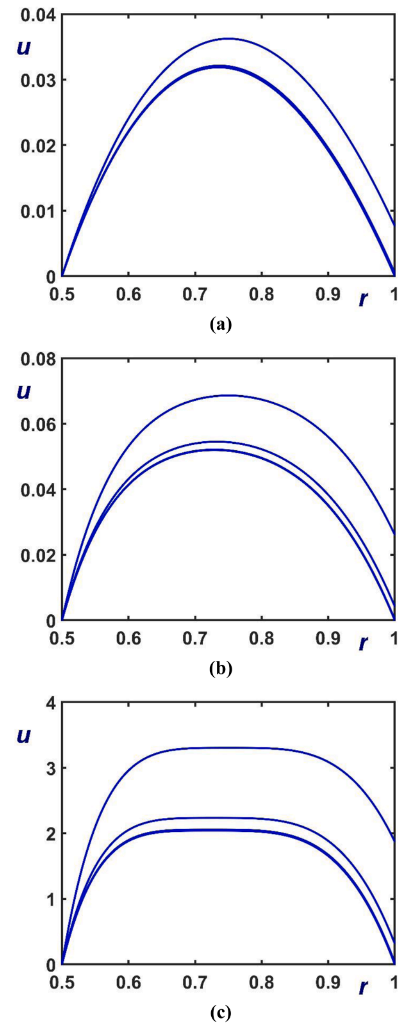

Fig. 6. Convergence of the method for a linear PTT fluid with $\kappa = 0.5$ and $\varepsilon = 0.1$: (a) $De = 1$; (b) $De = 10$; (c) $De = 100$.

7. Concluding remarks

In this work, we have proposed a fast and accurate numerical scheme to determine the velocity profiles of generalised Newtonian, viscoplastic and PTT fluids in a concentric annulus under a constant pressure gradient. In this scheme the unknown constant $b$of Eq. (2) is calculated by means of a shooting method. The methods available in the literature do not address how the unknown constant b affects the numerical solution. Our algorithm faces this problem front on and delivers a solution to an accuracy of 10^{-10}.

We shall now turn to helical flows. This flow can be generated by the rotations of the inner and outer cylinders with two different constant angular velocities, and the fluid is pushed along the annular space with a constant applied pressure gradient. Obviously, the angular and axial velocity components of the velocity field have to be determined. Here, two difficulties arise. The first one is due to the shear rate given by

$$\dot{\gamma} = \left(r^2 \omega'^2 + u'^2\right)^{1/2} \tag{42}$$

where $\omega = \omega(r)$ is the angular velocity and $u = u(r)$ is the axial velocity, and the primes denote the respective derivatives. Except in a Newtonian fluid, the viscosity depends on the shear rate. Secondly, the shear stresses $\tau_{r\theta}$ and τ_{rz} are given by

$$\tau_{r\theta} = \eta(\dot{\gamma}) r\omega', \quad \tau_{rz} = \eta(\dot{\gamma}) u' \tag{43}$$

Table 4
Convergence of the method for a linear PTT fluid with $\kappa = 0.1$ and $\varepsilon = 0.1$.

De	Iterations	Calculated R	Theoretical R
1	7	0.45854479761240	0.45854479761240
10	9	0.40407770745614	0.40407770745614
100	9	0.39443913497519	0.39443913497502

Table 5
Convergence of the method for a linear PTT fluid with $\kappa = 0.5$ and $\varepsilon = 0.1$.

De	Iterations	Calculated R	Theoretical R
1	5	0.73538190465439	0.73538190465436
10	6	0.72896672909371	0.72896672909371
100	6	0.72405445723366	0.72405445723357

It is well known that the equations of motion lead to the following shear stress distributions:

$$\tau_{r\theta} = \frac{M}{2\pi r^2}, \quad \tau_{rz} = \frac{b}{r} - \frac{r}{2}, \quad R_1 \le r \le R_2 \tag{44}$$

Here, M is the non-dimensional external torque per unit length to maintain the angular motion of the fluid. So, any numerical scheme has to deal with the ubiquitous constant b, as well as the shear stresses coupled through the viscosity function. At present, there is no known

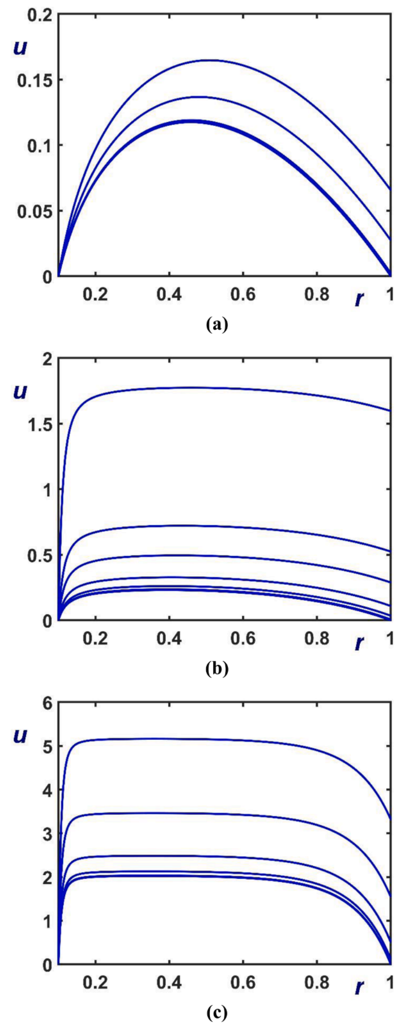

Fig. 7. Convergence of the method for an exponential PTT fluid with $\kappa = \varepsilon = 0.1$: (a) $De = 1$; (b) $De = 5$; (c) $De = 10$.

efficient numerical scheme to solve this problem. Even in the case of the Bingham fluid, an algorithm does not exist because the location of the yield surfaces, if any, depends on determining the region where $\dot{\gamma} = 0$. For a thorough discussion of this matter, see [12].

The helical flow can also be produced when the inner cylinder moves axially and both cylinders rotate. In this case, there is no need for a pressure gradient in the annulus. While the velocity field can be found analytically for both Newtonian and Bingham fluids [12,13], the determination of the velocity field for generalised Newtonian and other fluids requires a numerical scheme.

It can now be seen that the algorithm presented here can be extended to apply to helical flows. First of all, we determine $u = u(r)$ as if the flow is rectilinear. Secondly, we determine $\omega = \omega(r)$ as if the axial flow does not exist. For instance, one can let $\omega(\kappa) = \Omega_1, \quad \omega(1) = \Omega_2$. These two velocity fields lead to a new value of the combined shear rate through Eq. (42). In turn, this can be used to find updated values of and $u = u(r)$. The iterations continue till $u(1) = 0$ to within an acceptable error term.

Declaration of Competing Interest

The Authors declare that there is no conflict of interest.

References

[1] R.B. Bird, W.E. S.tewart, L.N. Lightfoot, Transport Phenomena, Wiley, New York, 1960.
[2] A.G. Fredrickson, R.B. Bird, Non-Newtonian flow in annuli, Ind. Eng. Chem. 50 (1968) 347–352.
[3] R.W. Hanks, The axial laminar flow of yield-pseudoplastic fluids in a concentric annulus, Ind. Eng. Chem. Process Des. Dev. 18 (1979) 488–493.
[4] J. David, Z. Beran, P. Filip, Explicit solution of laminar axial flow of power-law fluids in concentric annuli, Acta Tech. CSAV 37 (1993) 725–734.
[5] F.T. Pinho, P.J. Oliveira, Axial annular flow of a nonlinear viscoelastic fluid - an analytical solution, J. Non-Newtonian Fluid Mech. 93 (2000) 325–337.
[6] W. Herschel, R. Bulkley, Measurement of consistency as applied to rubber-benzene solutions, Proc. Am. Soc. Test Mater. 26 (1926) 621–633.
[7] N. Phan-Thien, R.I. Tanner, A new constitutive equation derived from network theory, J. Non-Newtonian Fluid Mech. 2 (1977) 353–365.
[8] N. Phan-Thien, A nonlinear network viscoelastic model, J. Rheol. 22 (1978) 259–283.
[9] T.C. Papanastasiou, G. Georgiou, A. Alexandrou, Viscous Fluid Flow, CRC Press, Boca Raton, 2000.
[10] R.B. Bird, R.C. Armstrong, O. Hassager, Dynamics of Polymeric Liquids, 2nd Ed., 1, John Wiley and Sons, New York, 1987.
[11] M. Chatzimina, C. Xenophontos, G. Georgiou, I. Argyropaidas, E. Mitsoulis, Cessation of annular Poiseuille flows of Bingham plastics, J. Non-Newtonian Fluid Mech. 142 (2007) 135–142.
[12] R.R. Huilgol, Fluid Mechanics of Viscoplasticity, Springer, New York, 2015.
[13] J. Peng, K.-.Q. Zhu, Linear stability of Bingham fluids in spiral Couette flow, J. Fluid Mech. 512 (2004) 21–45.