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J.G. Oldroyd's early ideas leading to the modern understanding of wall slip

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ABSTRACT

In this paper J.G. Oldroyd's ideas leading to the modern understanding of wall slip phenomena are discussed. The procedure proposed by Oldroyd to analyze experimental data that imply the presence of slip is illustrated to (i) determine the slip velocity as a function of wall shear stress and (ii) to recover the true rheological parameters of the material in the absence of slip. The Oldroyd method is demonstrated by using capillary data for a Thermoplastic Vulcanizate (TPV) polymer melt and a bread dough.

1. Introduction

Wall slip has been considered by several scientists in the early stages of the development of Newtonian fluid mechanics including Bernoulli, Coulomb, Poiseuille, Girard, Maxwell, and Navier (see [1] for a summary). However, several classes of complex fluids exhibit phenomena inconsistent with the assumption of no-slip, including polymer melts [2–4], elastomers [5], polymer solutions [6–9], suspensions [10–12], dispersions [13–14], gels [15–17], colloidal dispersions/glasses [18], pastes [19] and foams [20–22]. The study of wall slip is extremely important as it can be used to determine the true rheology of fluids by correcting the data for slip effects and explaining the mismatch of rheological data obtained from various rheometers utilizing different geometries. Interfacial (slip) constitutive laws are also needed to simulate these flows either from a macroscopic, microscopic and/or molecular point of view [23–25].

This article discusses the early work on slip of non-Newtonian fluids with particular emphasis on Oldroyd's ideas on slip and his influence on our modern understanding of slip both from the mathematical and experimental perspectives [26]. Inspired by this work, we perform some additional calculations to illustrate several issues addressed particularly for the case of yield stress fluids, i.e., how slip effects may manifest themselves when the flow curves are determined in simple shear and capillary flow.

The rest of the paper is organized as follows: The early literature on wall slip is reviewed in Section 2 along with Oldroyd's slip method. This method is derived in Section 3. In Section 4 the application of Oldroyd's method is demonstrated, by analysing capillary flow data for two cases (i) a thermoplastic vulcanizate polymer melt and (ii) a bread dough, to obtain both the rheological and wall slip parameters. Concluding remarks are finally provided in Section 5.

2. Early work on slip of non-Newtonian fluids

Apart from the consideration of slip effects for the case of Newtonian fluids in the 19th century which is summarized in Ref. [1], significant work has been reported in the 1930′s for the case of complex fluids, which is the year of beginning of the Journal of Rheology. Reiner [27] proved mathematically that if the fluidity (defined below) is plotted against the stress at the wall, and the flow curve is independent of the dimensions of the apparatus, irrespective of the flow law of the liquid under test, the no-slip boundary condition should apply. Consequently, Reiner [28] studied the capillary flow data of a 1.71% solution of nitro-cotton in di-butyl phthalate, which is a non-Newtonian liquid [29]. Using his method, he concluded that this fluid appears to slide on a glass wall. Slip appeared to increase with the age of the solution.

Schofield and Scott Blair [30–34] and Mooney [35] studied slip in capillary flow and demonstrated that the flow curve of soil pastes exhibits slip effects or marked discrepancies at the wall. They concluded that these discrepancies can be accounted for by assuming that in the immediate proximity of the wall a modification of the plastic properties occurs, which imparts an additional velocity (slip) to the bulk of the material. By first subtracting this (slip) velocity, a viscosity constant is obtained independent of the dimensions of the tube. Schofield and Scott Blair derived explicit formulae to calculate the slip velocity from capillary experimental data [31]. In a subsequent paper [32], they

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defined a depletion layer (very thin compared to the capillary diameter – approximated up to 10% of the diameter) close to the wall, where they concluded that the consistency (viscosity) of the material (soil pastes) should be much different than that in the bulk. At a later time, Schofield and Scott Blair [33] performed detailed measurements of the rate of flow of an aqueous paste of barium sulfate through tubes of various lengths and diameters in an attempt to study the influence of yield stress on slip. Their results showed that the rate of flow is independent of the length of the tube but depended on the capillary diameter, concluding that apparent slip persists at the proximity of the wall. Simple relations to address and explain discrepancies of flow curves due to the slip at the wall were derived by Schofield [34].

Mooney [35] derived explicitly formulae that can be used to analyze conveniently rheological data from capillary, and rotating cylinder geometries to calculate the wall slip as a function of wall shear stress (still used today!!!). Experimental data reported by Schofield and Blair [30, 36] was analysed and demonstrated the robustness and validity of the Mooney method. Interestingly, most of these works [28–33,35,36] were published within a 2-year period (1930–1931) and as mentioned above are still used based on the same principles. For example, these methods have been used extensively for the case of polymers melts [25] polymer solutions [7–9] microgels [37], and other complex fluids [38,39].

2.1. Oldroyd's work on slip

Oldroyd [26] published a single theoretical paper on wall slip in 1949. Three cases were examined, that of a Newtonian, a shear thinning, and a Bingham fluid in Poiseuille flow. He has shown that a wall effect can be measured by an effective slip coefficient, which is a function of the wall stress. The mathematical argument is based on the following assumptions: (i) the flow is rectilinear with axial symmetry; (ii) there is no actual slip at the wall; and (iii) the functional relationship between stress and rate-of-strain is dependent on the normal distance from a solid boundary, in a thin layer of liquid near the boundary only. The last assumption is equivalent of stating that within the thin layer, the viscosity of a complex fluid changes due to particles (suspensions) or polymer chain migration (polymer solutions) due to configurational entropic effects.

Oldroyd's method is a variant of the pioneering analysis of Mooney [8,35], which in turn was based on the work of Schofield and Scott Blair [36] in the early 1930s. Much later in 1967, Jastrzebski [40] adjusted the principles introduced by Mooney and Oldroyd to accommodate experimental observations for pipe flows of certain slurries, where the apparent slip velocity was observed to depend not only on the wall shear stress but also on the pipe diameter. The Oldroyd-Jastrzebski method proved to be more appropriate than the classical Mooney method to slurries [41], foams [42–44] and food suspensions [45,46]. However, there is no physical reason to support the diameter dependence of the slip velocity. For this reason, the Mooney's variant methods have been the subject of criticism [47,48]. Mooney's procedure, for example, may lead to erroneous results, such as negative slopes and intercepts for certain complex fluids such as elastomers [5] and pastes [45,47,48]. Martin and Wilson [47] demonstrated that the use of the Jastrzebski interface condition yields incorrect flow curves and apparent slip velocity and recommending that the method it should no longer be used. Nevertheless, the method is applied successfully in studies of energized fluids and foams used in hydraulic fracturing [49]. Modified versions of these methods have also been proposed for highly concentrated non-Brownian suspensions [50].

3. The Oldroyd's slip method

3.1. Derivation of the method

In their analysis of steady-state axisymmetric Poiseuille flow of any fluid, Schofield and Scott Blair [30] noted that the velocity gradient in the radial direction can be written as follows:

$$\frac{du}{dr} = -2f(\tau(r)) = -2f\left(\frac{r\tau_w}{R}\right) \tag{1}$$

where R is the tube radius, f is an odd function, $\tau = |\tau_{rz}|$, $\tau_w = GR/2$ is the wall shear stress, and G is the imposed pressure gradient. (The factor 2 in Eq. (1) has been added in order to be consistent with Oldroyd's notation [26]). Eq. (1) is easily realized, noting that du/dr is the local shear rate for fully developed Poiseuille flow, that is obviously a function of the local shear stress. Integrating Eq. (1) under the assumption of slip, one gets

$$u(r) = u_w + 2\int_r^R f(\xi)d\xi \tag{2}$$

where u_w is the "effective" slip velocity. In the absence of slip, Eq. (2) is simplified to

$$u(r) = 2\int_r^R f(\xi)d\xi \tag{3}$$

Observing that $r = R\tau(r)/\tau_w$ and changing variables, Schofield and Scott Blair [30] expressed the velocity as a function of the shear stress:

$$u(\tau) = \frac{2R}{\tau_w}\int_\tau^{\tau_w} f(s)ds \tag{4}$$

Hence, they derived an expression for the volumetric flow rate as follows:

$$Q = 2\pi\int_0^R u(r)rdr = 2\pi\int_0^{\tau_w} u(\tau)\frac{R\tau}{\tau_w}\frac{Rd\tau}{\tau_w} = \frac{2\pi R^2}{\tau_w^2}\int_0^{\tau_w}\tau d\tau\cdot\frac{2R}{\tau_w}\int_\tau^{\tau_w} f(s)ds \;\Rightarrow\; Q = \frac{4\pi R^3}{\tau_w^3}\int_0^{\tau_w}\tau d\tau\int_\tau^{\tau_w} f(s)ds \tag{5}$$

The latter expression indicates that, for any material, $Q/(\pi R^3)$ depends only on the wall shear stress τ_w. After an integration by parts, Eq. (5) may also be written as follows

$$Q = \frac{2\pi R^3}{\tau_w^3}\int_0^{\tau_w}\tau^2 f(\tau)\,d\tau \tag{6}$$

In fact, Oldroyd [26] worked with the apparent fluidity, φ, as defined by Poiseuille's formula,

$$\varphi = \frac{4Q}{\pi R^3\tau_w} \tag{7}$$

Combining Eqs. (6) and (7) yields that the fluidity is a function of shear stress

$$\varphi(\tau_w) = \frac{8}{\tau_w^4}\int_0^{\tau_w}\tau^2 f(\tau)\,d\tau \tag{8}$$

or

$$\varphi(\tau_w) = F(1/\tau_w) \tag{9}$$

where

$$F(\xi) = 8\xi^4\int_0^{1/\xi} x^2 f(x)\,dx \tag{10}$$

Oldroyd [26] pointed out that if the curve $\varphi(\tau_w)$ is unique, then the shear rate at the wall can be calculated and the stress/shear rate relationship can be deduced by means of the function f, which can be extracted from experimental data. Indeed, one can solve for this function as follows:

$$F(1/\xi) = \frac{8}{\xi^4}\int_0^\xi x^2 f(x)dx \;\Rightarrow\; \xi^4 F(1/\xi) = 8\int_0^\xi x^2 f(x)dx \;\Rightarrow\; \frac{d[\xi^4 F(1/\xi)]}{d\xi} = 8\xi^2 f(\xi) \;\Rightarrow$$

$$f(\xi) = \frac{1}{8\xi^2}\frac{d[\xi^4 F(1/\xi)]}{d\xi} \tag{11}$$

Oldroyd [26] underlined that a ‘very sharply defined smooth curve’ is needed for the numerical differentiation of Eq. (11). He also acknowledged the fact that Rabinowitch and Mooney also derived a similar formula involving differentiation of a known functional relationship.

Oldroyd provided the functions f and F for Newtonian fluids, Bingham plastics and bi-viscosity fluids [26]. Moreover, he analysed the characteristics of the $(\varphi,\ 1/\tau_w)$ curve. In the case of a Herschel-Bulkley fluid described by

$$\begin{cases} \dot\gamma = 0, & \tau \le \tau_0 \\ \tau = \tau_0 + k\dot\gamma^n, & \tau > \tau_0 \end{cases} \tag{12}$$

where τ_0 is the yield stress, k is the consistency index, and n is the power law exponent, the functions f and F are as follows:

$$f(\xi) = \begin{cases} 0, & \xi \le \tau_0 \\ \frac{1}{2}\left(\frac{\xi-\tau_0}{k}\right)^{1/n}, & \xi > \tau_0 \end{cases} \tag{13}$$

and

$$F(\xi)=\begin{cases} 0, & \xi \ge 1/\tau_0 \\ \frac{4\xi^{1-1/n}}{(3+1/n)k^{1/n}}(1-\tau_0\xi)^{1/n}\left\{1-\frac{\tau_0\xi}{2n+1}\left[1+\frac{2n\tau_0\xi}{n+1}(1+n\tau_0\xi)\right]\right\}, & \xi<1/\tau_0 \end{cases} \tag{14}$$

3.2. Wall effects

Oldroyd [26] emphasized the importance of taking wall slip into account: “*… a single observed curve showing an increase of φ with $1/\tau_w$ at large τ_w may be a manifestation of a wall effect rather than an indication either that the viscosity increases with the rate of shear at large rates of shear or that the flow is non-laminar.*”

In the presence of slip the $(\varphi, 1/\tau_w)$ curve also depends on the tube radius. Mooney [35] and then Reiner [28] demonstrated that Eq. (9) is extended to

$$\varphi(\tau_w, R) = F(1/\tau_w) + \frac{4u_w}{R\tau_w} \tag{15}$$

where u_w is an effective slip velocity, which is a function of the wall shear stress. Oldroyd [26] proposed an alternative approach avoiding the assumption of an effective slip velocity. More specifically, he assumed that Eq. (1) holds in the bulk and that in a layer adjacent to the wall the thickness ε of which is independent of R, the rate of strain also depends on ε and on the distance from the wall, i.e.

$$\frac{du}{dr} = -2f\left(\frac{r\tau_w}{R}\right) + g\left(\frac{r\tau_w}{R}, R-r, \varepsilon\right), \quad R-\varepsilon \le r \le R \tag{16}$$

Integrating the two branches of the velocity and taking the no-slip condition into account leads to

$$Q = 2\pi\int_0^R r^2 f\left(\frac{r\tau_w}{R}\right)dr + 2\pi\int_{R-\varepsilon}^R r^2 g\left(\frac{r\tau_w}{R}, R-r, \varepsilon\right)dr \quad \Rightarrow$$

$$\varphi = \frac{4Q}{\pi R^3\tau_w} = \frac{8}{\tau_w^4}\int_0^{\tau_w}\xi^2 f(\xi)\,d\xi + G(\tau_w, R, \varepsilon)$$

or

$$\varphi(\tau_w, R, \varepsilon) = F(1/\tau_w) + G(\tau_w, R, \varepsilon) \tag{17}$$

Following Mooney [35], Oldroyd assumed that the wall layer thickness is very small compared to the tube radius ($\varepsilon \ll R$), in order to simplify the above expression:

$$\varphi(\tau_w, R) = F(1/\tau_w) + G(\tau_w, R) \tag{18}$$

By means of a change of variables, Oldroyd reduced Eq. (18) to the relation

$$\varphi(\tau_w, R) = F(1/\tau_w) + \frac{4\zeta(\tau_w)}{R} \tag{19}$$

where

$$\zeta(\tau_w) = \frac{2}{\tau_w}\int_0^{\varepsilon(\tau_w)} g[\varepsilon(\tau_w), \tau_w, x]\,dx \tag{20}$$

may be viewed as the effective slip coefficient. The advantage of Eq. (19) is that both F and ζ are independent of R. This is equivalent to Reiner’s formula (Eq. (15)) if the effective slip velocity satisfies

$$u_w = \tau_w\zeta(\tau_w) \tag{21}$$

It is clear that $(1/R, \varphi)$ data fall on a line of slope $4\zeta(\tau_w)$ intercepting the φ axis at $F(1/\tau_w)$ for a given value of τ_w. Oldroyd [26] pointed out that “*the limiting value of ζ at large τ_w may be supposed finite and is zero if u_w is bounded when τ_w increases*”. This is consistent with the experimental findings of Chakrabandhu and Singh [46] for coarse food suspensions at high temperatures.

As already noted, Oldroyd’s approach is a variant of Mooney’s method [35]. With the latter method, one plots instead $(1/R, \tau_w\varphi)$ data and thus the slope of the resulting line is $4u_w$ (see Eq. (15)). Jastrzebski [40] reported that for slurries the apparent slip velocity also depends on the pipe radius, i.e.

$$u_w = \frac{\tau_w\bar\zeta(\tau_w)}{R} \tag{22}$$

and, thus, in the Oldroyd-Jastrzebski method Eq. (19) is replaced by

$$\varphi(\tau_w, R) = F(1/\tau_w) + \frac{4\bar\zeta(\tau_w)}{R^2} \tag{23}$$

Finally, Crawford et al. [51] generalized Jastrzebski’s method introducing an adjustable exponent d, so that the slip velocity is given by

$$u_w = \frac{\tau_w\bar\zeta(\tau_w)}{R^d} \tag{24}$$

4. Application of Oldroyd’s method

Oldroyd’s method has been applied on capillary flow data for two cases (i) a Thermoplastic Vulcanizate (TPV melt) and (ii) wheat dough. The analysis of these experimental results to calculate the slip velocity and the true rheological parameters are presented next.

4.1. Capillary slip flow of a thermoplastic vulcanizate melt

Fig. 1 presents capillary flow data for a TPV melt at 190 °C, for two different capillary diameters: $D_1 = 0.889$mm, $D_2 = 0.432$mm[52,53]. The capillary length to diameter ratio (L/D) in both cases was 14. Only seven data points have been obtained in each case for shear rates ranging from 5 to 1000 s^{-1} (see Fig. 1). We will demonstrate now that using the Oldroyd method and this minimal set of experimental data plotted in Fig 1, we can recover the slip velocity and the true rheological parameters of the fluid to a certain extend.

The two data sets were fitted to power-law models of the form

$$\dot\gamma_i = a_i\tau_w^{b_i}, \quad i = 1, 2 \tag{25}$$

where the subscript stands for the number of the experiment, to obtain the optimal parameter values: $a_1 = 3.36\cdot 10^7 \mathrm{MPa}^{-b_1}\mathrm{s}^{-1}$, $b_1 = 5.03$, $a_2 = 2.21\cdot 10^7 \mathrm{MPa}^{-b_2}\mathrm{s}^{-1}$, and $b_2 = 4.44$. The fitted lines are compared with

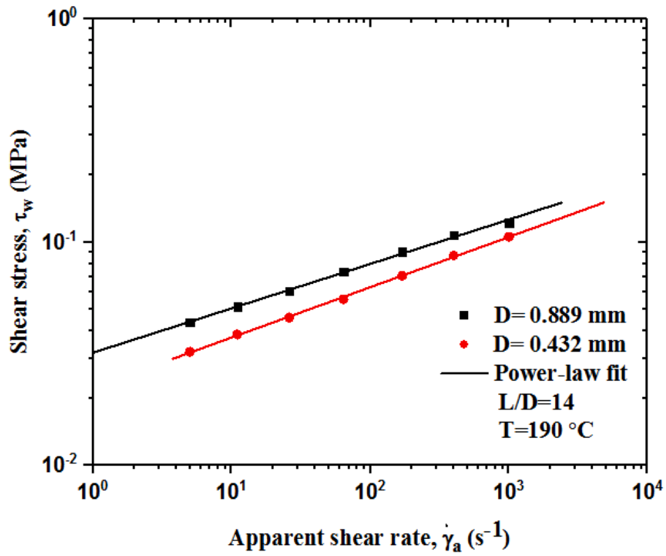

Fig. 1. Fitting of the experimental data on a TPV melt to Eq. (25).

the experimental data in Fig. 1.

A uniform distribution of wall shear stress values in the range of interest, i.e. for values of τ_w from 0.03 to 0.15 MPa, was then selected and the corresponding shear rates were calculated by means of Eq. (25). The apparent fluidity for each data set was found by means of

$$\varphi_i(\tau_w, R_i) = \frac{\dot{\gamma}_i}{\tau_w} = \frac{a_i \tau_w^{b_i}}{\tau_w} = a_i \tau_w^{b_i - 1}, \quad i = 1, 2 \tag{26}$$

Since the fluidity is known for the two values of the radius, both $F(1/\tau_w)$ and $\zeta(\tau_w)$ can be calculated from Eq. (19) for all the selected values of the wall shear stress τ_w:

$$F(1/\tau_w) = \frac{a_1 R_1 \tau_w^{b_1-1} - a_2 R_2 \tau_w^{b_2-1}}{R_1 - R_2} \tag{27}$$

and

$$\zeta(\tau_w) = \frac{R_1 R_2}{4(R_1 - R_2)} \left(a_2 \tau_w^{b_2-1} - a_1 \tau_w^{b_1-1} \right) \tag{28}$$

The corresponding slip velocities are then calculated by means of Eq. (21):

$$u_w = \frac{R_1 R_2}{4(R_1 - R_2)} \left(a_2 \tau_w^{b_2} - a_1 \tau_w^{b_1} \right) \tag{29}$$

Slip data are nicely fitted to a power-law slip equation

$$\tau_w = \beta u_w^s \tag{30}$$

yielding $\beta = 0.039\text{MPas}^s/\text{mm}^s = 0.00735\text{MPas}^s/\text{m}^s$ and $s = 0.24$. The predictions of Eq. (30) essentially coincide with the points calculated by means of Eq. (29), indicating that the power-law slip model describes very well the slip data in the range of the experimental wall shear stresses. These are compared with the data points reported by Ghahramani et al. [53] in Fig. 2, where the slip velocity u_w is measured in mm/s. Note that the dashed line labelled as Ghahramani et al. [52,53] has utilized also parallel plate data at lower wall shear stresses (Fig. 2). The Oldroyd method predicted the experimental slip data remarkably well.

Since F is known (given by Eq. (27)), it is straightforward to obtain the function $f(\xi)$ by means of Eq. (11):

$$f(\xi) = \frac{1}{8(R_1 - R_2)} \left[a_1 R_1 (b_1 + 3) \xi^{b_1} - a_2 R_2 (b_2 + 3) \xi^{b_2} \right] \tag{31}$$

It is clear that the above expression holds only if $f(\xi) \geq 0$. Therefore, it

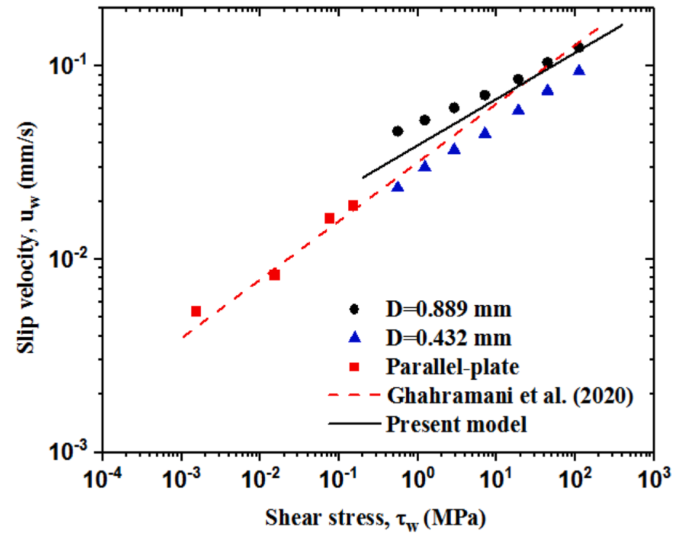

Fig. 2. Comparison of the calculations of the slip Eq. (30) (Oldroyd method) shown as a continuous solid line with the slip experimental calculations of Ghahramani et al. [53]. The dashed line labelled as "present model" is the line passing through all experimental points including those from parallel plate (simple shear) calculated by Ghahramani et al. [53]. The agreement is remarkable.

should be $\xi \geq \xi^*$, where

$$\xi^* = \left[\frac{a_2 R_2 (b_2 + 3)}{a_1 R_1 (b_1 + 3)} \right]^{1/(b_1 - b_2)} = 0.126\text{MPa} \tag{32}$$

provides a first estimate of the yield stress τ_0. Essentially the same estimate is obtained by fitting the predictions of Eq. (31) to the Herschel-Bulkley model (13), which also gives $k = 1.335 \times 10^{-4}\text{MPas}^n$ for the consistency index and $n = 0.77$ for the power-law exponent. The predictions of the fitted model are compared with the points calculated by means of Eq. (31) in Fig. 3. It is interesting to note that slip brings down the flow curve even well below the yield stress level. This implies that even if the fluid does not yield for stresses below its true yield stress, it can slip as a purely elastic solid. The predicted flow curve is compared with the experimental data in Fig. 4. While the true yield stress of this material is 0.09 MPa obtained from linear viscoelastic measurements

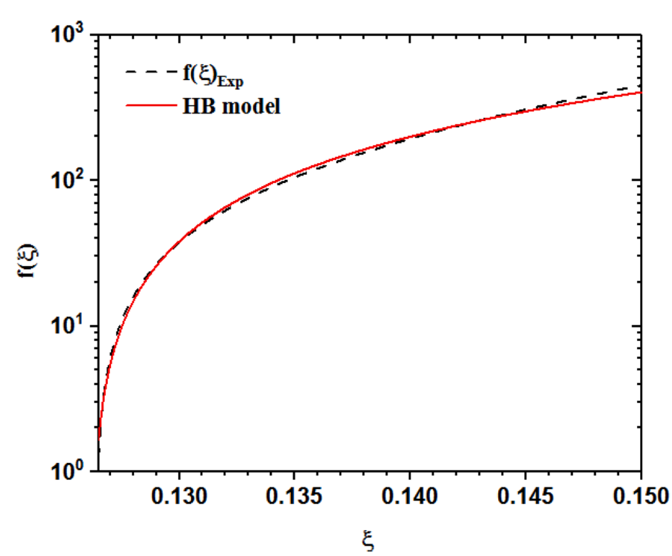

Fig. 3. Fitting the function $f(\xi)$ for the Herschel-Bulkley (HB) model to the predictions of Eq. (31).

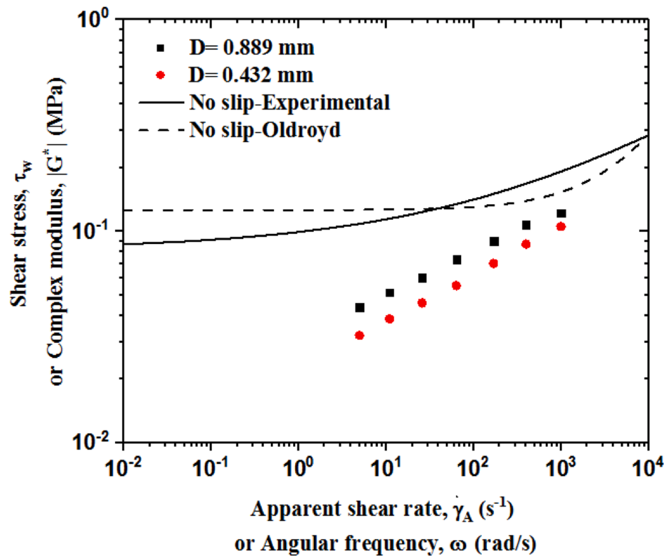

Fig. 4. The predicted flow curve using the Oldroyd's method (dashed line) compared with the experimental data corrected for the effect of slip (continuous line).

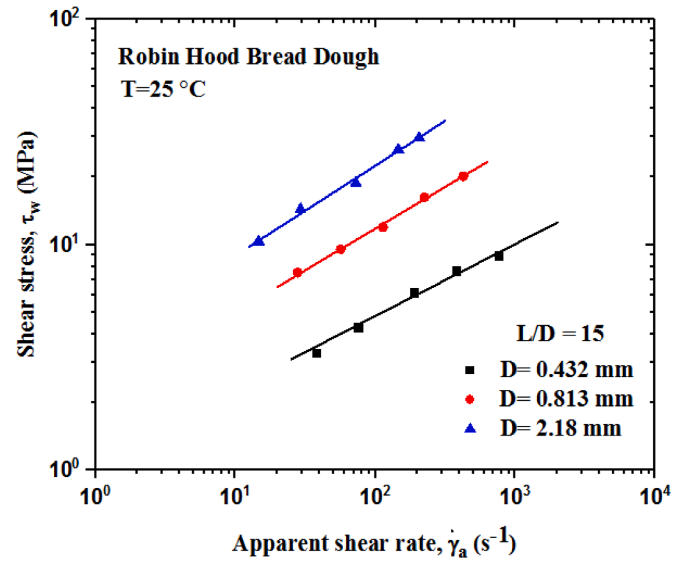

Fig. 5. The Bagley-corrected apparent flow curve of Robin Hood Bread Dough obtained from dies of various diameters at 25 °C. The diameter dependence of these flow curves indicate that slip occurs at the wall [54].

[52,53], the value calculated by means of Oldroyd's analysis is 0.126 MPa, which is about 35% higher. However, considering the use of only two sets of capillary data (capillary data inherently possess a reproducibility of $\pm$10%) of different diameter without the application of the Bagley correction (assumed small), the recovery of the Herschel-Bulkley parameters (comparison in Fig. 4) may be considered quite satisfactory. Moreover, applying the Oldroyd's method has resulted in slip velocity estimates, which agree well with those obtained with the more elaborate slip analysis by Ghahramani et al. [52,53].

As a concluding remark for the slip flow of yield stress fluids that follow the Herschel-Bulkley rheological law and others, one has to consider two regimes, the unyielded regime, where $\tau_w \le \tau_0$, and the yielded regime, where $\tau_w > \tau_0$. In the former regime, the velocity is flat, the fluid slips as a purely elastic solid i.e. $u_z(r) = u_w$, and thus $\dot{\gamma}_A = 4u_w/R$. Hence, the slip velocity can be calculated from

$$u_{wi} = \frac{\dot{\gamma}_{A,i} R_i}{4}, \quad i = 1,2 \tag{33}$$

By means of slip Eq. (30) one gets

$$\dot{\gamma}_{A,i} = \frac{4}{R_i}\left(\frac{\tau_w}{\beta}\right)^{1/s}, \quad \tau_w \le \tau_0 \tag{34}$$

In the yielded regime, the apparent shear rate is given by

$$\dot{\gamma}_{A,i} = \frac{4}{R_i}\left(\frac{\tau_w}{\beta}\right)^{1/s} + \frac{4}{3+1/n}\left(\frac{\tau_w-\tau_0}{k}\right)^{1/n} \times \left\{1 - \frac{1}{(2n+1)}\frac{\tau_0}{\tau_w}\left[1 + \frac{2n}{(n+1)}\frac{\tau_0}{\tau_w}\left(1 + n\frac{\tau_0}{\tau_w}\right)\right]\right\}, \quad \tau_w > \tau_0 \tag{35}$$

The above expression can be useful in analyzing capillary experimental data for yield stress fluids.

4.2. Capillary flow of bread dough

Fig. 5 depicts capillary flow data of a bread dough using three sets of capillary dies of different diameters at 25 °C reported by Sofou et al. [54]. The diameter dependence of the flow curve implies the presence of slip. Sofou et al. [54] analyzed the data and found that the Mooney analysis cannot be applied as it yields negative true shear rates after correcting the data for slip effects. Instead they used a modified Mooney method proposed by Geiger [55] that assumes the slip velocity is a function of the die diameter, an assumption that has no physical basis. In this section we reanalyze these data by means of the Oldroyd method.

First, it is noted that typical bread doughs possess particles sizes as small as 200 mm (0.2 mm) [56] whose dimensions are similar to the smallest diameter used in the experiments (0.432 mm). Therefore, this set of data should be excluded from the analysis. For multiphase systems it is essential to consider the depletion layer formed at the wall, which the Oldroyd method includes by considering its thickness ε. Thus, the effective diameter to perform the analysis of capillary data is $D_{eff} = D - 2\varepsilon$. We consider in this analysis various values of εaround 300 μm (0.3 mm), that is 1.5 times the particle size (a reasonable assumption) to match the experimental results [54]. The results are summarized in Table 1 and are also plotted in Fig. 6 for three selected values of ε that best describe the experimental data [54]. It seems that a value of $\varepsilon = 0.29 - 0.30$ mm best matches the experimental results, resulting also in a yield stress of 0.243–0.398 kPa that includes the experimental value of 0.298 kPa reported in [54]. In conclusion, the Oldroyd method (i) yields the true rheological parameters reported in [54], (ii) calculates the slip velocity in the form of $\tau_w = \beta u_w^s$ with slip velocity independent of diameter (the exponent s is independent of the depletion layer thickness ε), and (iii) gives an estimate of the depletion layer consistent with the particle size.

5. Conclusions

In this paper, J.G. Oldroyd's ideas leading to the modern understanding of wall slip phenomena have been discussed. Specifically, we have discussed and illustrated the procedure proposed by Oldroyd to analyze experimental data that imply the presence of slip. The Oldroyd's method can help to determine the slip velocity as a function of wall shear stress and moreover recover the true rheological parameters of the material in the absence of slip (true rheological response of the fluid under consideration). The method is demonstrated by using capillary data for two different systems, namely a Thermoplastic Vulcanizate (TPV) polymer melt and a bread dough. The recovered rheological parameters were compared with those obtained from linear viscoelastic measurements (absence of slip) and found to be in good agreement. In the case of multiphase systems such as the bread dough, the Oldroyd method also yields a good estimate of the depletion layer thickness to match the experimental results and a slip velocity law that is

Table 1
Estimates of the slip ($\tau_w = \beta u_w^s$) and rheological parameters of Herschel-Bulkley ($\tau_w = \tau_o + k\dot{\gamma}^n$) for various values of the depletion layer thickness ε.

Depletion Thicknessε, *mm*	Slip coefficientβ, kPas^s/m^s	Slip exponent*s*	Yield stressτ_o, kPa	Consistency index*k*, kPas^n	Power law exponent,*n*
0.28	100.8	0.372	0.621	4.27	0.585
0.29	104.5	0.372	0.398	4.20	0.509
0.30	108.5	0.372	0.243	4.13	0.470
0.31	113.1	0.372	0.165	4.01	0.449
0.32	118.4	0.372	0.109	3.90	0.435
0.35	140.8	0.372	0.029	3.60	0.412

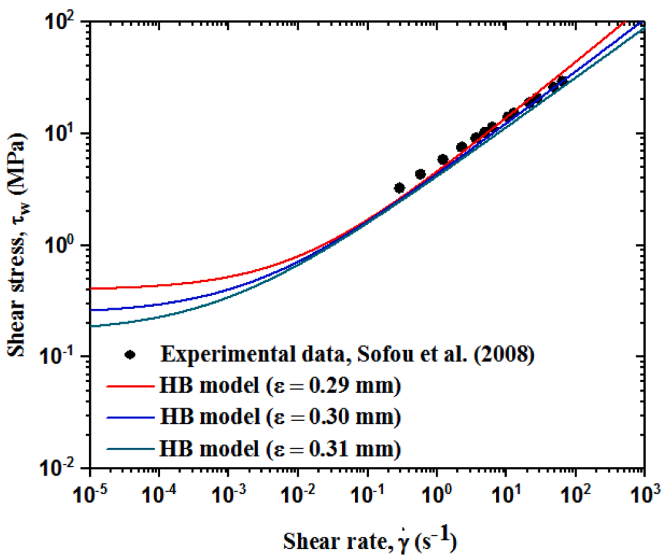

Fig. 6. Comparison of the constitutive equations obtained with $\varepsilon = 0.29,\ 0.30,$ 0.31 mm with the experimental data of Sofou et al. [54].

independent of the die diameter. Finally, expressions are provided for slip analysis of capillary rheological data of Herschel-Bulkley fluids based on Oldroyd's method.

Declaration of Competing Interest

The authors declare that there is no conflict of interest.

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