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Cheeger sets and the minimum pressure gradient problem for viscoplastic fluids

G.C. Georgiou [a,*], R.R. Huilgol [b]

[a] Department of Mathematics, University of Cyprus, PO Box 20537, 1678 Nicosia, Cyprus
[b] College of Science and Engineering, Flinders University, GPO Box 2100, Adelaide SA 5001, Australia

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ABSTRACT

A Cheeger set of a domain, Ω, is a subset of this domain such that the ratio of its perimeter to its area, h, is a minimum, if such a subset exists, with the possibility that the subset may be the original domain as well. This value h is called the Cheeger constant for a given domain. If one considers the reciprocal of this minimum or, the maximum ratio of the area of the subset to its perimeter, $M = 1/h$, it follows from the work of Mosolov and Miasnikov that the minimum pressure gradient, G, to sustain a steady flow in a pipe with a cross-section defined by Ω is given by $G > \tau_y/M$, where τ_y is the constant yield stress of a viscoplastic fluid. Using the results of Kawohl and Lachand-Robert, we derive the Cheeger constant for a square in two different ways. The application of their results to a convex polygon including a triangle, which leads to a different method to find the relevant Cheeger constant, and rotationally symmetric cross-sections are also described. Finally, a new method to determine the Cheeger constant for an ellipse is given.

1. Introduction

The Cheeger set Ω_c of a domain Ω in a plane is the one that minimises the ratio of its perimeter to its area among all subsets D of Ω. It arose in the work of Cheeger [1] who proved a lower bound to for the smallest eigenvalue for the Laplacian under Dirichlet conditions in 1960. If Ω is convex, the Cheeger set exists and is unique [2,3]. It touches the boundary, although not everywhere. In places where Ω_c does not touch the boundary $\partial\Omega$, its free boundary $\partial\Omega_c \cap \Omega$ has constant curvature, or it is the arc of a circle. Moreover, this free boundary touches the boundary of Ω tangentially. Hence, to determine the Cheeger set of a given domain, it suffices to consider those subsets whose free boundary consists of arcs with circular arcs. For example, see the various figures in [4] and Fig. 9.1 in [5]. The ratio of the perimeter to the area of Ω_c, denoted by h, is called the *Cheeger constant* of Ω.

Independently, in 1965, Mosolov and Miasnikov [6] proved that the minimum pressure gradient, G_c, for a steady flow of a Bingham fluid to exist in a pipe of a given cross-section was determined by the maximum ratio of the area of a subset of the domain to its perimeter. This value, M, is clearly given by $M = 1/h$. In addition, they also proved [6] that $G_c > \tau_y/M$, where τ_y is the yield stress of the Bingham fluid. This result is explained in full in §2.

Hence, to solve the minimum pressure gradient problem, it suffices to determine the Cheeger constant of the domain defining the cross-section of a pipe. While numerous publications have appeared to determine the Cheeger constant since 1960, we restrict our attention to convex cross-sections only and employ the results of Kawohl and Lachand-Robert [2]. In particular, the survey article by Parini [7] as well as the more recent work by Cañete [8] reinforce the importance of the results in [2]. To highlight this work, we appeal to two theorems in [2] to determine the Cheeger constant for a square in two different ways. Extensions of the method to find the Cheeger constant for a convex polygon including a triangle, and that for rotationally symmetric cross-sections [8] are listed. Our new result in this work is to find the Cheeger constant for an ellipse, based on Fig. 7 in [4]. These matters are fully explained in Sections 3–5.

Since Mosolov and Miasnikov [6] did not make the connection of their results with Cheeger sets, their important role in the flows of Bingham fluids was unknown till Frigaard et al. [9] showed that critical yield numbers for particles to settle in Bingham fluids depended on Cheeger sets. In the work reported here, we have been influenced by the results of the settling problem to emphasise the connection between Cheeger sets and the minimum pressure gradient problem for all incompressible viscoplastic fluids, with a constant yield stress.

* Corresponding author.
E-mail addresses: georgios@ucy.ac.cy (G.C. Georgiou), raj.huilgol@flinders.edu.au (R.R. Huilgol).

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2. The minimum pressure drop per unit length to initiate a steady flow

Consider a pipe of arbitrary cross-section defined through Ω in (x, y) coordinates, with its boundary defined by $\partial\Omega$. Suppose that a steady axial flow of an incompressible viscoplastic fluid exists in this pipe with the velocity field defined through

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \dot{z} = w(x, y) \geq 0, \quad w(x, y)\Big|_{\partial\Omega} = 0. \tag{2.1}$$

This flow is assumed to occur under a constant pressure drop, i.e., $\partial p/\partial z = -G,\ G > 0$. The components of the relevant Rivlin–Ericksen tensor [10] for the velocity field in Eq. (2.1) are given by

$$A_{13} = A_{31} = w_{,x} \quad A_{23} = A_{32} = w_{,y}, \tag{2.2}$$

where the commas denote the respective partial derivatives. It is also known that $A_{11} = A_{22} = A_{33} = A_{12} = A_{21} = 0$.

Hence, only two shear stresses exist in the yielded region. They are

$$S_{xz} = S_{zx} = \eta(|\nabla w|)w_{,x} + \frac{\tau_y}{|\nabla w|}w_{,x}, \quad S_{yz} = S_{zy} = \eta(|\nabla w|)w_{,y} + \frac{\tau_y}{|\nabla w|}w_{,y}, \tag{2.3}$$

since

$$II(\mathbf{A}) = |\nabla w|, \tag{2.4}$$

and

$$\nabla w = w_{,x}\mathbf{i} + w_{,y}\mathbf{j}. \tag{2.5}$$

In Eq. (2.3), $\eta(|\nabla w|)$ is the shear rate dependent viscosity, and τ_y is the constant yield stress of a viscoplastic fluid. Obviously, shear stresses exist in the unyielded regions as well and in them, they obey the inequality:

$$0 \leq II(\mathbf{S}) = \left[S_{xz}^2 + S_{yz}^2\right]^{1/2} \leq \tau_y. \tag{2.6}$$

The main problem for the flow in a pipe of arbitrary cross-section may now be posed: Is there a minimum pressure drop per unit length to initiate the flow? The answer to this question can be found from Lemmas 2.2 and 2.3 proved by Mosolov and Miasnikov [6]. In order to apply these Lemmas, one has to begin with the energy equation for the flow of a viscoplastic fluid in a pipe of arbitrary cross-section. This energy equation for the steady flow of such a fluid is given by:

$$\frac{1}{2}\int_\Omega \eta(|\nabla w|)|\nabla w|^2\,da + \tau_y\int_\Omega |\nabla w|\,da = G\int_\Omega w\,da. \tag{2.7}$$

The basic idea is to turn the above into an inequality by replacing the right side by an upper bound. In order to achieve this, two Lemmas will now be stated from the work of Mosolov and Miasnikov [6], reproduced from [4].

Lemma 1. *If $h(x, y)$ is a smooth function satisfying the condition*

$$h(x, y)\Big|_{\partial\Omega} = 0, \tag{2.8}$$

then

$$M\int_\Omega |\nabla h|\,da \geq \int_\Omega h\,da, \quad M = \sup_{\Omega' \subseteq \Omega} \frac{A(\Omega')}{P(\Omega')}. \tag{2.9}$$

Here, Ω' is an arbitrary sub-domain of Ω with a boundary $\partial\Omega'$. Next, $A(\Omega')$ is the area of this sub-domain and $P(\Omega')$ is its perimeter. One notes that there is no restriction that Ω should possess a symmetric cross-section. This Lemma can be applied to the right side of Eq. (2.7) and results in the following inequality:

$$\frac{1}{2}\int_\Omega \eta(|\nabla w|)|\nabla w|^2\,da + \tau_y\int_\Omega |\nabla w|\,da \leq MG\int_\Omega |\nabla w|\,da. \tag{2.10}$$

Obviously, there will be an infinite number of sub-domains of Ω. Unless one finds a specific sub-domain which delivers the bound M, the above Lemma is not of much use. The following Lemma provides the clue regarding the shape of this sub-domain.

Lemma 2. *There exists a sub-domain Ω_c with boundary $\partial\Omega_c$ for which*

$$M = \frac{A(\Omega_c)}{P(\Omega_c)}, \tag{2.11}$$

where, if Q is a point on $\partial\Omega_c$ not on $\partial\Omega$, then the connected part of the set $\partial\Omega_c\backslash\partial\Omega$, containing Q, is the arc of a circle touching $\partial\Omega$.

This says in essence that if a point Q lies on the boundary $\partial\Omega_c$ and not on the boundary $\partial\Omega$, then the arc on which Q lies is circular and this arc is tangential to $\partial\Omega$ when it meets it. The procedure to find M relies on this observation. Note that the given domain Ω may, in itself, be the optimal one as in the case of a circular disk. Otherwise, an interior one exists which furnishes M.

This constant M is important, for it determines the minimum pressure gradient to initiate the flow of a viscoplastic fluid in a pipe. To see this, one rewrites Eq. (2.10) as:

$$\frac{1}{2}\int_\Omega \eta(|\nabla w|)|\nabla w|^2\,da \leq (MG - \tau_y)\int_\Omega |\nabla w|\,da. \tag{2.12}$$

If $MG - \tau_y \leq 0$, it follows that $|\nabla w|^2 \leq 0$, since the viscosity $\eta(|\nabla w|) > 0$. That is, the flow must have a constant velocity across the cross-section of the pipe. However, the velocity field $w = 0$ on the boundary of the pipe, which means that $w = 0$ in the pipe. Hence, we conclude that a steady flow of the viscoplastic fluid will exist in a pipe provided the pressure drop per unit length G satisfies the following inequality:

$$G > \frac{\tau_y}{M} \tag{2.13}$$

Thus, to find this value of G, one must determine M for a given cross-section.

If the cross-section of the pipe is symmetrical, such as that of a square, a rectangle or an equilateral triangle, the task of finding M can be reduced to a simple optimisation problem. That is, one finds the radius of the circular arc which rounds off the corners at the vertices. This simple procedure was employed by Huilgol [4]; it has been reproduced with some further elucidation in the monograph by Huilgol and Georgiou [5].

The foregoing discussion raises several questions:

1. Does the set Ω_c exist for all cross-sections?
2. Is this set unique?
3. If the set Ω is that of a convex polygon and Ω_c exists, does its boundary $\partial\Omega_c$ touch every side of this polygon?
4. If the set Ω is convex and rotationally symmetric, can one find Ω_c?
5. Are there other methods to determine the constant M?

These questions lead us to Cheeger sets which are considered next.

3. Cheeger sets

The set Ω_c is also known as a Cheeger set. It arose in the work of Cheeger [1] who proved a lower bound for the smallest eigenvalue, λ_1, for the Laplacian under Dirichlet boundary conditions. Let h be defined through

$$h = \inf_{\mathscr{C}(\Omega)\subseteq\Omega} \frac{P(\mathscr{C}(\Omega))}{A(\mathscr{C}(\Omega))}, \tag{3.1}$$

provided it exists. Here, $P(\mathscr{C}(\Omega))$ is the perimeter and $A(\mathscr{C}(\Omega))$ is the area of the set $\mathscr{C}(\Omega)$. In the sequel, whenever the Cheeger set exists, we shall denote it by Ω_c.

The number, h, in Eq. (3.1) is called the *Cheeger constant*. It is clearly the reciprocal of the constant M, defined in Eq. $(2.9)_2$. Cheeger's result [1] that $\lambda_1 \geq h^2/4$ is not of relevance to the contents of this paper, whereas the constant h is crucial. To be specific, when Ω is convex, it is known that the Cheeger set Ω_c exists. It is unique and is the union of a set of disks of radius $1/h$ [11]. Clearly, this result lies behind the applications of the Mosolov–Miasnikov Lemmas [4–6].

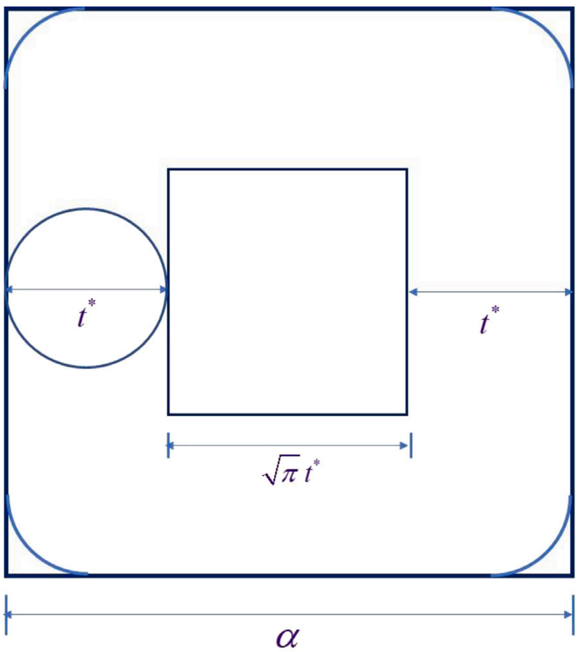

Fig. 1. The sets Ω^{t^*} and $t^* B_1$ for a square.

Hence, the determination of h for a given convex set is of importance here.

While various methods have been proposed to determine h, it appears that the theorems proved by Kawohl and Lachand-Robert [2] provide the simplest tools. As mentioned earlier, the importance of these theorems has been reinforced in the works Parini [7] and Cañete [8]. We state these theorems to explain their applications to determine the Cheeger sets for specific convex cross-sections in the plane, beginning with Theorem 1.

For any given convex set Ω in the plane, we denote by $\mathrm{dist}(x, \partial\Omega)$ the distance from $x \in \Omega$ to a point on the boundary $\partial\Omega$. For any $t \geq 0$, we denote the points of distance at least t by

$$\Omega^t := \{x \in \Omega | \;\; \mathrm{dist}(x, \partial\Omega) > t\}. \tag{3.2}$$

So, the boundary $\partial\Omega^t$ is the inner parallel set to $\partial\Omega$ at a distance t at least. We have:

Theorem 1. *There exists a unique value $t = t^* > 0$ such that $|\Omega^t| = \pi t^2$. Then, $h(\Omega) = 1/t^*$ and the Cheeger set of Ω is $\Omega_c = \Omega^{t^*} + t^* B_1$, with B_1 denoting the unit disk.*

That is, the set Ω^{t^*} is parallel to the set Ω and lies inside of it, at a distance of t^* at least from the boundary $\partial\Omega$. As an example, we apply Theorem 1 to the case when Ω is a square, with a side of length a. This derivation is different from that in [4–6] where the constant M was found through Eq. $(2.9)_2$.

It is easy to see that the set Ω^{t^*} is also a square, with its sides parallel to the boundary $\partial\Omega$ and placed symmetrically inside. This inner square has a side of length $\sqrt{\pi}\, t^*$. The disk $t^* B_1$ has a diameter of length t^*. Hence, from Fig. 2, we see that the Cheeger set Ω_c is the union of Ω^{t^*} and the set generated by the union of a disk of diameter t^* touching the boundary $\partial\Omega$, rounding off at the corners. It is essential to realise that the Cheeger set touches each side of the square. In Theorem 3, we list the necessary and sufficient conditions for the Cheeger set to touch each side of a convex polygon.

Returning to Fig. 1, we see quite easily that

$$(2 + \sqrt{\pi})t^* = a. \tag{3.3}$$

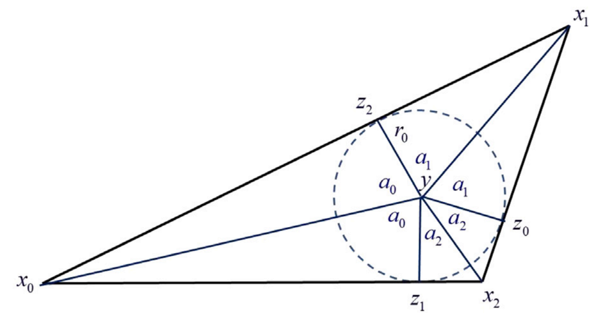

Fig. 2. Cheeger set for a triangle.

It is known from the work of Mosolov–Miasnikov that the value of $t^* = a/(2 + \sqrt{\pi})$; see [4–6]. The calculation of t^* from Theorem 1 is much more direct, however.

If the domain Ω is convex and the curvature of its boundary is not finite, such as in the case of an ovoid, Theorem 1 can be applied to find its Cheeger set. See Fig. 1 in [2].

Suppose that the set Ω^{t^*} is difficult to find for a given convex domain Ω. Here, Theorem 2 from [2] is helpful in some cases. This is stated as:

Theorem 2. *Let Ω be any convex set, $\bar{\kappa}$ the maximum value of its curvature. Then $\Omega_c = \Omega$ if and only if*

$$\bar{\kappa}|\Omega| \leq |\partial\Omega|. \tag{3.4}$$

That is the Cheeger set of Ω is itself provided the product of $\bar{\kappa}$ and its area, $|\Omega|$, is less than or equal to its perimeter $|\partial\Omega|$. This theorem can be applied to find the Cheeger constant h for an ellipse as follows. First of all, one observes that the boundary of an ellipse does not have a constant curvature, varying along its boundary. Indeed, let the ellipse have a semi-major axis of length a and a semi-minor axis of length $b < a$. The maximum curvature occurs at the end points of the major axis and is given by $\bar{\kappa} = a/b^2$.

For an ellipse, the area has the simple result $|\Omega| = \pi ab$. Its eccentricity k is given by $k = \sqrt{a^2 - b^2}/a$. The perimeter $|\partial\Omega|$ is given by

$$|\partial\Omega| = 4a \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \xi}\, d\xi, \quad 0 \leq k < 1. \tag{3.5}$$

This integral is an elliptical integral of the second kind:

$$E(k, \pi/2) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \xi}\, d\xi. \tag{3.6}$$

Since $b = a\sqrt{1 - k^2}$, one finds from the foregoing that if

$$E(k, \pi/2) \geq \frac{\pi}{4\sqrt{1 - k^2}}, \tag{3.7}$$

the Cheeger set of the ellipse is itself, leading to $h = |\partial\Omega|/|\Omega|$. Numerical computations of $E(k, \pi/2)$ for various values of k can be found in [5] and one finds that Eq. (3.7) holds provided $0 \leq k < 0.79117$, or $(b/a) > 0.6116$, confirming the result in [2]. In Section 5, we shall derive the shape of the Cheeger set for an arbitrary ellipse.

4. General convex polygons

Suppose that a polygon is convex, in general; it need not be symmetric. For such a domain, Theorem 3 in [2], quoted below, provides the required tools to find the Cheeger constant.

Let the convex polygon have n sides, with vertices at $x_0, x_1, \ldots, x_{n-1}$, $x_n = x_0$, ordered counter-clockwise. Let the interior angle at x_i be given by $\pi - 2\alpha_i$. We note that $\alpha_i \in (0, \pi/2)$ because the polygon is convex.

Let $l_i = |x_i - x_{i-1}|$ be the length of the ith side, so that $|\partial\Omega| = \sum_{i=1}^{n} l_i$. Define the sum of the tangents of the angles $\alpha_i,\ i = 1,\dots,n$, through

$$T(\Omega) = \sum_{i=1}^{n} \tan\alpha_i. \tag{4.1}$$

Since $\sum_{i=1}^{n} \alpha_i = \pi$, it follows that $T(\Omega) > \pi$ because for any $x \in (0, \pi/2)$ we have $\tan x > x$. With these preliminaries, we can ask whether the Cheeger set of the polygon touches every side of it, which is called a *Cheeger-regular* set. The following Theorem is proved in by Kawohl and Lachand-Robert [2].

Theorem 3. *A convex polygon Ω is Cheeger-regular if and only if*

$$|\Omega| - r_0|\partial\Omega| + r_0^2(T(\Omega) - \pi) \le 0, \tag{4.2}$$

$$r_0 = \min_{1\le i\le n} \frac{l_i}{\tan\alpha_i + \tan\alpha_{i-1}}. \tag{4.3}$$

In this case, the perimeter and area of the Cheeger-regular set are given respectively by

$$|\partial\Omega_c| = |\partial\Omega| - 2(T(\Omega) - \pi)r, \tag{4.4}$$

$$|\Omega_c| = |\Omega| - (T(\Omega) - \pi)r^2 = r|\partial\mathscr{C}_\Omega|. \tag{4.5}$$

Here, r is the smaller root of

$$(T(\Omega) - \pi)r^2 - r|\partial\Omega| + |\Omega| = 0, \tag{4.6}$$

whence,

$$r = |\partial\Omega| - \sqrt{|\partial\Omega|^2 - 4(T(\Omega) - \pi)|\Omega|}. \tag{4.7}$$

Thus, the Cheeger constant has the following simple formula:

$$h(\Omega) = \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 - 4(T(\Omega) - \pi)|\Omega|}}{2|\Omega|}. \tag{4.8}$$

As an example, consider the case when the polygon is a square, with a side of length a each. Here, $T(\Omega) = 4,\ r_0 = a/2$. The left side of Eq. (4.2) is $(-\pi a^2/4) < 0$. Thus, Eq. (4.8) leads to the following result:

$$h = \frac{2 + \sqrt{\pi}}{a}, \tag{4.9}$$

which is the same as in Eq. (3.3).

In order to find h through Eq. (4.8), the only difficult task is to determine $T(\Omega)$ for a given polygon, since $|\partial\Omega|$ and $|\Omega|$ are quite easy to find. For instance, when Ω is a triangle, the three bisectors of the included angles meet at a point. Using this as the centre, an inscribed circle can be drawn touching all the three sides. See Fig. 2, which also appears as Fig. 4 in [2]. It shows that the radius of this circle is r_0, and, from Eq. (4.3), it follows that r_0 is the same for each $i = 1, 2, 3$. It is also quite easy to see from the six small triangles that $|\Omega| = r_0|\partial\Omega|/2$. Next,

$$|\partial\Omega| = \sum_{i=0}^{2} r_0(\tan\alpha_i + \tan\alpha_{i+1}) = 2r_0T(\Omega). \tag{4.10}$$

Hence, for any triangle,

$$h(\Omega) = \frac{|\partial\Omega| + \sqrt{4\pi|\Omega|}}{2|\Omega|}. \tag{4.11}$$

For an equilateral triangle of side a each, it is obvious that $|\partial\Omega| = 3a,\ |\Omega| = \sqrt{3}\,a^2/4$. Hence, Eq. (4.11) provides the following result:

$$h(\Omega) = \frac{2\left(3 + \sqrt{\pi\sqrt{3}}\right)}{\sqrt{3}\,a}. \tag{4.12}$$

This value has been determined earlier by Huilgol [4] through the Mosolov-Miasnikov Lemmas.

It has been proved recently by Cañete [8] that Cheeger constants for rotationally symmetric convex plane figures can be found from Theorem 1 in [2]. Moreover, such polygons are Cheeger-regular, meaning that the Cheeger set touches each side of such a polygon.

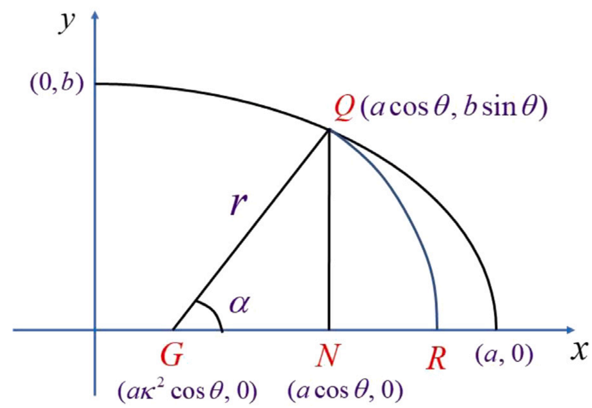

Fig. 3. Relevant points of the ellipse in the first quadrant.

If the domain Ω is not Cheeger-regular, i.e., the Cheeger set does not touch each side of a given convex polygon, an algorithm has been proven by Kawohl and Lachand-Robert [2] to find $h(\Omega)$. We do not pursue this matter here.

5. Cheeger set for an ellipse

Let the equation of an ellipse be given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0. \tag{5.1}$$

Referring to Fig. 3, which is based on Fig. 7 in [4], we consider a point Q on the ellipse in the first quadrant. Here, it is simpler to work in terms of the eccentric angle θ [12]. The coordinates of the point Q are $(a\cos\theta, b\sin\theta)$. The area under the ellipse in $0 \le x \le a\cos\theta$ is given by

$$E(\theta) = \frac{ab}{4}\int_\theta^{\pi/2} \sin^2\xi\, d\xi = \frac{ab}{4}\Big[\pi - 2\theta + \sin 2\theta\Big]. \tag{5.2}$$

The length $L(\theta)$ of the arc is given by

$$\begin{aligned} L(\theta) &= \int_\theta^{\pi/2} \sqrt{(dx/dt)^2 + (dy/dt)^2}\; dt \\ &= \int_\theta^{\pi/2} \sqrt{a^2\sin^2 t + b^2 cos^2 t}\; dt \\ &= a\int_0^{\pi/2-\theta} \sqrt{1 - k^2\sin^2\xi}\; d\xi, \end{aligned} \tag{5.3}$$

where k is the eccentricity of the ellipse. The integral in Eq. (5.3) is the elliptic integral of the second kind. MATLAB, for example, can be used to determine it.

The normal to the ellipse through the point Q, with the coordinates $(a\cos\theta, b\sin\theta)$, intersects the x-axis at the point G with its coordinates given by $(ak^2\cos\theta, 0)$ [12], provided $0 < \theta \le \pi/2$. Let us denote the point with the coordinates $(a\cos\theta, 0)$ by N. Hence, the points (G, N, Q) form a right-angled triangle. See Fig. 3 again. The area of the triangle is given by

$$T(\theta) = \frac{1}{2}(a\cos\theta - ak^2\cos\theta)b\sin\theta = \frac{ab\sin 2\theta(1 - k^2)}{4}. \tag{5.4}$$

Let the angle formed by the line GQ with the x-axis be given by α. It is easy to see that

$$\tan\alpha = \frac{b\sin\theta}{(a\cos\theta - ak^2\cos\theta)} = \frac{b}{a(1 - k^2)}\tan\theta. \tag{5.5}$$

Next, the area of the sector passing through Q with G as its centre and meeting the x-axis at $R = (ak^2\cos\theta + r, 0)$ is given by $r^2\alpha/2$, where the radius r is the length of the line GQ. Note that we are using the property of the Cheeger set that the free boundary must be the arc of a circle, tangential to the boundary of the ellipse. Thus

$$\begin{aligned} r^2 &= a^2\cos^2\theta(1-k^2)^2 + b^2\sin^2\theta \\ &= a^2(1-k^2)[1-k^2\cos^2\theta]. \end{aligned} \tag{5.6}$$

Hence, the area of a candidate for the Cheeger set in the first quadrant is given by

$$A(\theta) = E(\theta) - T(\theta) + \frac{1}{2}r^2\alpha, \tag{5.7}$$

which also takes the following form

$$A(\theta) = \frac{ab}{4}[\pi - 2\theta + k^2\sin 2\theta] + \frac{b^2}{2}\left(1-k^2\cos^2\theta\right)\tan^{-1}\left(\frac{a}{b}\tan\theta\right). \tag{5.8}$$

Its perimeter is the sum of the arc from $(0,b)$ to Q along the ellipse, plus that of the circular arc. This is given by

$$P(\theta) = L(\theta) + r\alpha, \tag{5.9}$$

which yields the following:

$$P(\theta) = a\int_0^{\pi/2-\theta}\sqrt{1-k^2\sin^2\xi}\;d\xi + b\sqrt{1-k^2\cos^2\theta}\;\tan^{-1}\left(\frac{a}{b}\tan\theta\right). \tag{5.10}$$

So, given a and b, one has to find the minimum of $F(\theta) = P(\theta)/A(\theta)$, in order to determine the location of the optimal point Q on the ellipse. This is straightforward to compute numerically by finding the root of $F'(\theta) = 0$, or, equivalently, the root of $A'(\theta)P(\theta) - A(\theta)P'(\theta) = 0$ in $(0,\pi/2)$. The Cheeger sets for $a = 1$ and $b = 1/30$, $1/10$, $1/3$ and 0.45 are plotted in Figs. 4–7, where x_N and x_R denote the computed x-coordinates of the points N and R respectively.

If $k < 0.79117$, or $b/a > 0.6116$, it has been proven by Kawohl and Lachand-Robert [2] that the Cheeger set of the ellipse is the ellipse itself, as mentioned earlier. This can be seen in Fig. 8, where $b = 0.611$ and in Fig. 9, where $b = 0.6115$. In this connection, it is interesting to compare the computed value of h when $b/a = 0.6116$, with that obtained by Ramanujan's approximate formula for the perimeter of an ellipse. For an ellipse,

$$\frac{P(\Omega)}{A(\Omega)} = \frac{4a\int_0^{\pi/2}\sqrt{1-1-(1-b^2/a^2)\sin^2\xi}\,d\xi}{\pi ab}. \tag{5.11}$$

Hence, the Cheeger constant for an ellipse, when the Cheeger set is the ellipse itself, is given by

$$h = \frac{|\partial\Omega_c|}{|\Omega_c|} = \frac{4a\int_0^{\pi/2}\sqrt{1-1-(1-b^2/a^2)\sin^2\xi}\,d\xi}{\pi ab}. \tag{5.12}$$

Letting $a = 1, b = 0.6116$, one can compute the value of h using MATLAB and obtain

$$h = 2.67345908. \tag{5.13}$$

The famous formula of Ramanujan for the perimeter of the ellipse is:

$$P(\Omega) \approx \pi\left[3(a+b) - \sqrt{(3a+b)(a+3b)}\right]. \tag{5.14}$$

From this, letting $a = 1, b = 0.6116$, we find that

$$h = 2.67345804, \tag{5.15}$$

which is remarkably close to the value in Eq. (5.13) above.

Finally, we list the computed values of the Cheeger constants for the ellipses depicted in Figs. 4–9, in Table 1.

A second method of optimisation to find h is to let the coordinates of the point Q on the ellipse be given by (x,y). Here, one optimises the location of the point Q through the determination of its x-coordinate [13]. We have found it easier to compute the optimal value of θ in order to determine the Cheeger constant.

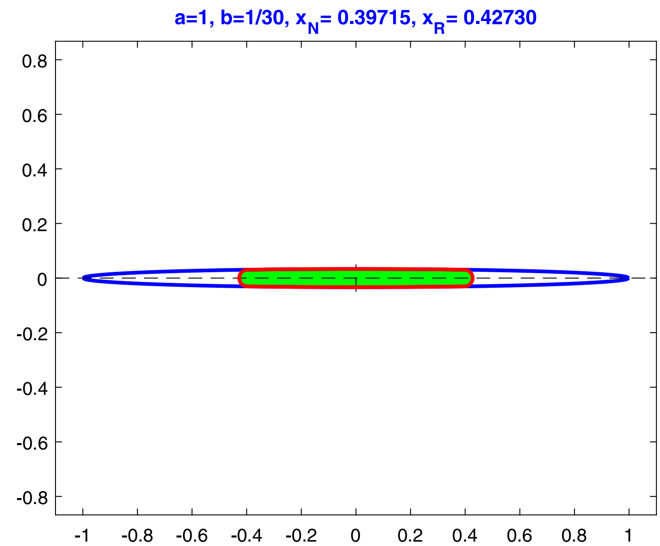

Fig. 4. Cheeger set of an ellipse with $a = 1$, $b = 1/30$.

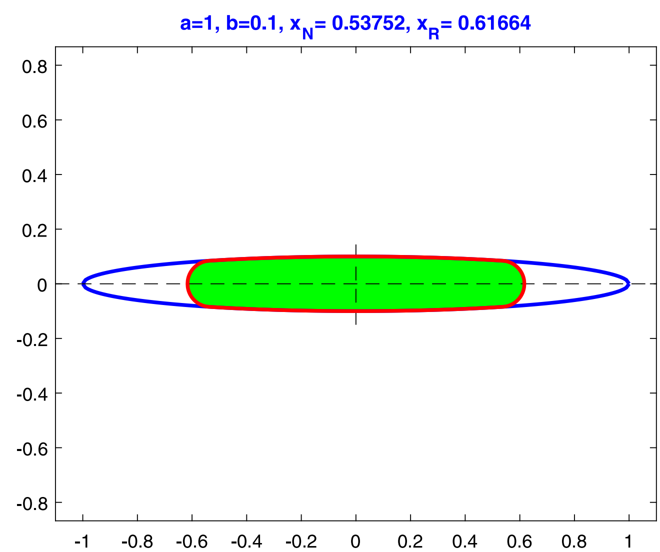

Fig. 5. Cheeger set of an ellipse with $a = 1$, $b = 0.1$.

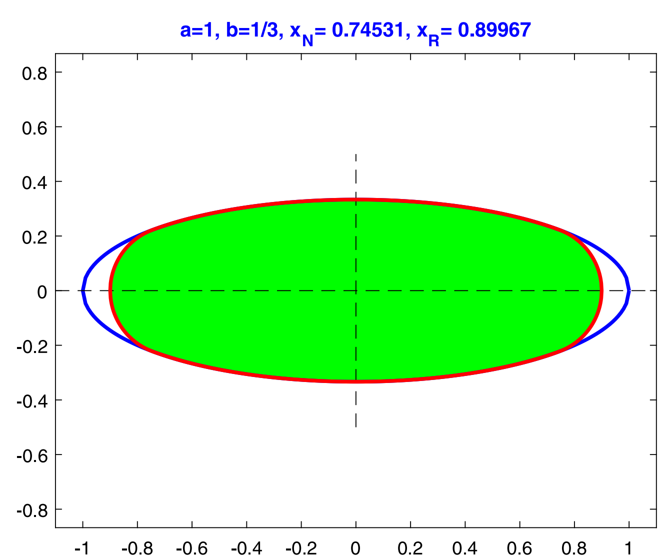

Fig. 6. Cheeger set of an ellipse with $a = 1$, $b = 1/3$.

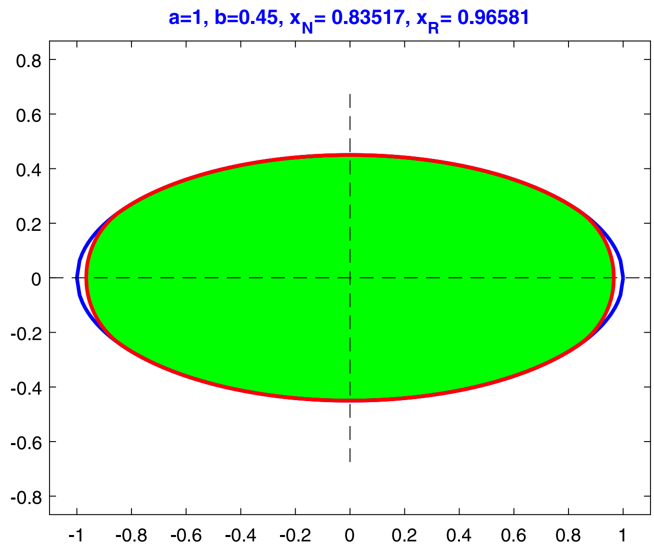

Fig. 7. Cheeger set of an ellipse with $a = 1$, $b = 0.45$.

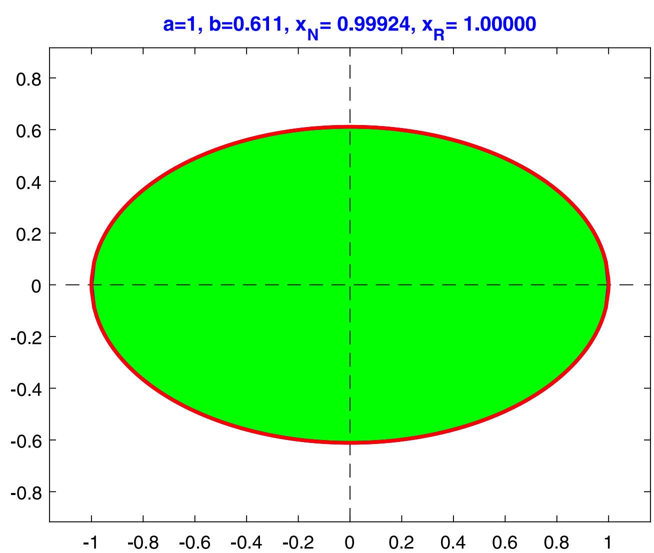

Fig. 8. Cheeger set of an ellipse with $a = 1$, $b = 0.611$.

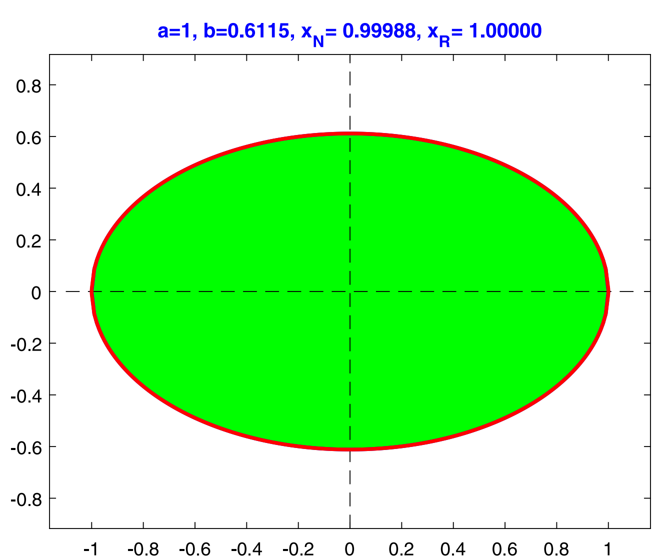

Fig. 9. Cheeger set of an ellipse with $a = 1$, $b = 0.6115$.

Table 1
Calculated Cheeger constants for different values of b, when $a = 1$.

b	h
1/30	32.69657
0.1	11.83780
1/3	4.21534
0.45	3.33451
0.611	2.67524
0.6115	2.67376

6. Concluding remarks

Recognising the importance of the Cheeger constant in determining the minimum pressure gradient to sustain the steady flow of a Bingham fluid in pipes of convex cross-section, we have been able to re-derive the result when the cross-section is a square [4,6]. In addition, we have reproduced the results, when the cross-section is an ovoid, or a convex polygon including a triangle, from the three theorems proved by Kawohl and Lachand-Robert [2]. Finally, the Cheeger constant when the cross-section is an ellipse has also been computed numerically for several values of b, when $a = 1$.

While some results for the elliptical cross-section appear in the recent monograph by Huilgol and Georgiou [5], the work presented here is complete and should be regarded as the final version.

If the cross-section of a pipe is not convex, such as a barbell-domain, the Cheeger set is not unique [2]. Similarly, there exist L-shaped domains which admit infinitely many Cheeger sets according to Parini [7]. While the minimum pressure gradient for a class of L-shaped cross-sections has been found in [4,5], the problem is open in general. For additional work on non-convex domains, see the Remarks and the references listed by Cañete [8].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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