Bergman Polynomials on Archipelaga: Recent Developments on Theory and Applications

To the memory of Herbert Stahl

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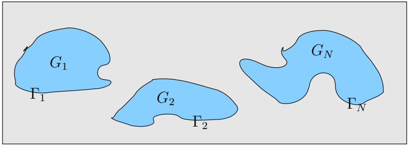
CMFT 2013
Shantou, China
June 2013

Bergman polynomials on an archipelago



Γ_j, $j = 1, \ldots, N$, a system of disjoint and mutually exterior Jordan curves in $\mathbb{C}$, $\boxed{G_j := \mathrm{int}(\Gamma_j)}$, $\boxed{\Gamma := \cup_{j=1}^{N} \Gamma_j}$, $\boxed{G := \cup_{j=1}^{N} G_j}$.

$$\langle f, g \rangle_G := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle_G^{1/2}$$

The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the unique orthonormal polynomials w.r.t. the area measure on G:

$$\langle p_m, p_n \rangle_G = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

Construction of the Bergman polynomials

Algorithm: Monomial Gram-Schmidt (GS)

Apply the Gram-Schmidt process to the monomials

$$1, z, z^2, z^3, \ldots$$

Main ingredient: the moments

$$\mu_{m,k} := \langle z^m, z^k \rangle_G = \int_G z^m \overline{z}^k dA(z), \quad m, k = 0, 1, \ldots .$$

The above algorithm has been been suggested, and used in practise, by pioneers of Numerical Conformal Mapping, like P. Davis and D. Gaier, in the 1960's, as the standard procedure for constructing Bergman polynomials. It was still treated as the standard method by leaders of the subject, like P. Henrici (*Computational Complex Analysis*, Vol. I,II and III), in the 1980's.

Instability Indicator

The GS method is notorious for its instability. For measuring it, when orthonormalizing a system $S_n := \{u_0, u_1, \ldots, u_n\}$ of functions, in a Hilbert space with norm $\|\cdot\|$, the following instability indicator has been proposed by J.M. Taylor, (Proc. R.S. Edin., 1978):

$$\boxed{I_n := \frac{\|u_n\|^2}{\min_{u \in \operatorname{span}(S_{n-1})} \|u_n - u\|^2}, \quad n \in \mathbb{N}.}$$

Note that, when S_n is an orthonormal system, then $\boxed{I_n = 1}$. When S_n is linearly dependent then $\boxed{I_n = \infty}$. Also, if $G_n := [\langle u_m, u_k \rangle]_{m,k=0}^n$, denotes the Gram matrix associated with S_n then,

$$\boxed{\kappa(G_n) \geq I_n},$$

where $\kappa(G_n) := \|G_n\| \|G_n^{-1}\|$ is the spectral condition number of G_n.

Instability of the Monomial GS process

Theorem

Let G be an archipelago and consider the application of the GS process to the monomials $S_n = \{1, z, z^2, \ldots, z^n\}$. If the boundary Γ of G satisfies an interior cone condition at the point z_0, where $|z_0| = \max\{|z| : z \in \Gamma\}$, then

$$\boxed{c_1(\Gamma)\, L^{2n} \leq I_n \leq \; c_2(\Gamma)\, L^{2n}},$$

where,

$$\boxed{L := \frac{\max\{|z| : z \in \Gamma\}}{\text{cap}(\Gamma)}} \quad (\geq 1).$$

Note that I_n is sensitive to the relative position of G w.r.t. the origin. Also $L = 1$, iff $G \equiv \mathbb{D}_r$. When G is the 8×2 rectangle centered at the origin, hen $L = 3/\sqrt{2} \approx 2.12$. In this case, $I_{25} \asymp 10^{16}$ and the method breaks down in MATLAB or FORTRAN, for $n = 25$.

The Arnoldi algorithm for OP's

Let μ be a (non-trivial) finite Borel measure with compact support supp(μ) on $\mathbb{C}$ and consider the associated series of orthonormal polynomials

$$p_n(\mu, z) := \lambda_n(\mu) z^n + \cdots, \quad \lambda_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

generated by the inner product

$$\langle f, g \rangle_\mu = \int f(z)\overline{g(z)} d\mu(z), \quad \|f\|_{L^2(\mu)} := \langle f, g \rangle_\mu^{1/2}.$$

Arnoldi GS for Orthonormal Polynomials

At the *n*-th step, apply GS to orthonormalize the polynomial zp_{n-1} (instead of z^n) against the (already computed) orthonormal polynomials $\{p_0, p_1, \ldots, p_{n-1}\}$.

Used by Gragg & Reichel, in Linear Algebra Appl. (1987), for the construction of Szegö polynomials.

Stability of the Arnoldi GS

Theorem (St, Constr. Approx (2013))

In the case of the Arnoldi GS, the instability indicator I_n satisfies

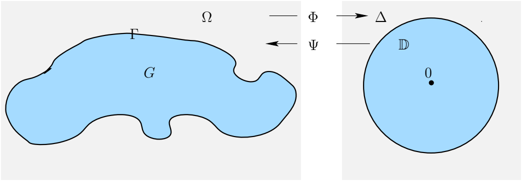
$$1 \leq I_n \leq \|z\|_{L^\infty(\mathrm{supp}(\mu))} \frac{\lambda_n^2(\mu)}{\lambda_{n-1}^2(\mu)}, \quad n \in \mathbb{N}.$$

- In the case of Szegö polynomials, where $d\mu \equiv |dz|$ on a system of smooth Jordan curves, we have

$$\boxed{c_1(\mu) \leq \frac{\lambda_n(\mu)}{\lambda_{n-1}(\mu)} \leq c_2(\mu)}, \quad n \in \mathbb{N}.$$

- When $d\mu \equiv w(x)dx$ on $[a, b] \subset \mathbb{R}$, then this ratio tends to a constant.

Asymptotics: Single-component case



$$\Omega := \overline{\mathbb{C}} \setminus \overline{G}$$

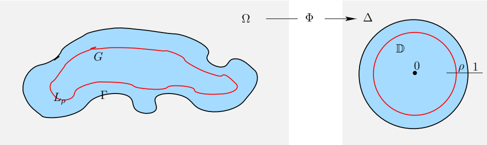
$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots . \qquad \text{cap}(\Gamma) = 1/\gamma$$

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots . \qquad \text{cap}(\Gamma) = b$$

The Bergman polynomials of G:

$$p_n(z) = \lambda_n z^n + \cdots , \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots .$$

Strong asymptotics when Γ is analytic



T. Carleman, Ark. Mat. Astr. Fys. (1922)

If $\rho < 1$ is the smallest index for which Φ is conformal in $\mathrm{ext}(L_\rho)$, then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\rho^{2n},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}}, \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \le c_2(\Gamma)\sqrt{n}\,\rho^n, \quad z \in \overline{\Omega}.$$

Strong asymptotics when Γ is smooth

We say that $\Gamma \in C(p, \alpha)$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$, if Γ is given by $z = g(s)$, where s is the arclength, with $g^{(p)} \in \mathrm{Lip}\alpha$. Then both Φ and $\Psi := \Phi^{-1}$ are p times continuously differentiable in $\overline{\Omega} \setminus \{\infty\}$ and $\overline{\Delta} \setminus \{\infty\}$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \mathrm{Lip}\alpha$.

P.K. Suetin, Proc. Steklov Inst. Math. AMS (1974)

Assume that $\Gamma \in C(p+1, \alpha)$, with $p + \alpha > 1/2$. Then, for $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,} \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\frac{1}{n^{2(p+\alpha)}},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}},$$

where

$$|A_n(z)| \le c_2(\Gamma)\,\frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}.$$

Strong asymptotics for Γ non-smooth

Theorem (St, C. R. Acad. Sci. Paris (2010) & Constr. Approx. (2013))

Assume that Γ is piecewise analytic without cusps. Then, for $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \textit{where} \quad 0 \le \alpha_n \le c(\Gamma)\,\frac{1}{n},$$

and for any $z \in \Omega$,

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + A_n(z)\right\}},$$

where

$$|A_n(z)| \le \frac{c_1(\Gamma)}{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

Ratio asymptotics for λ_n

Corollary (St, C. R. Acad. Sci. Paris (2010) & Constr. Approx. (2013))

$$\boxed{\sqrt{\frac{n+1}{n}}\,\frac{\lambda_{n-1}}{\lambda_n} = \mathrm{cap}(\Gamma) + \xi_n}, \quad \text{where} \quad |\xi_n| \le c(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}.$$

The above relation provides the means for computing approximations to the capacity of Γ, by using only the leading coefficients of the Bergman polynomials. In addition, it implies for the Instability Indicator I_n of the Arnoldi GS:

Corollary

$$\boxed{c_1(\Gamma) \le I_n \le c_2(\Gamma)}, \quad n \in \mathbb{N}.$$

Hence, the Arnoldi GS for Bergman polynomials, in the single component case, is stable.

Ratio asymptotics for $p_n(z)$

Corollary (St, C. R. Acad. Sci. Paris (2010) & Constr. Approx. (2013))

For any $z \in \Omega$, and sufficiently large $n \in \mathbb{N}$,

$$\boxed{\sqrt{\frac{n}{n+1}}\,\frac{p_n(z)}{p_{n-1}(z)} = \Phi(z)\{1 + B_n(z)\}},$$

where

$$|B_n(z)| \leq \frac{c_1(\Gamma)}{\sqrt{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

The above relation, combined with the Arnoldi GS, provides an efficient method for computing approximations to $\Phi : \Omega \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Note: The kernel polynomials $K_n(z, z_0) := \sum_{j=0}^{n} \overline{p_j(z_0)} p_j(z)$ are used in the Bergman kernel method for computing approximations to the interior conformal map $\varphi : G \to \mathbb{D}$.

Example: Computing the capacity of a Square

n	$\sqrt{\frac{n+1}{n}\frac{\lambda_{n-1}}{\lambda_n}}$	$\sqrt{\frac{n+1}{n}\frac{\lambda_{n-1}}{\lambda_n}} - \mathrm{cap}(\Gamma)$	s
100	0.834 640 612	1.37e-05	-
110	0.834 638 233	1.14e-05	1.9902
120	0.834 636 420	9.58e-06	1.9911
130	0.834 635 009	8.16e-06	1.9918
140	0.834 633 888	7.04e-06	1.9924
150	0.834 632 982	6.14e-06	1.9930
160	0.834 632 341	5.39e-06	1.9934
170	0.834 631 626	4.78e-06	1.9938
180	0.834 631 111	4.26e-06	1.9942
190	0.834 630 674	3.83e-06	1.9945
200	0.834 630 301	3.46e-06	1.9949

Γ: Square with corners at $1, i, -1, -i$. $\mathrm{cap}(\Gamma) = 0.834\,626\,841 \cdots$.

s: tests the hypothesis $\sqrt{\frac{n+1}{n}\frac{\lambda_{n-1}}{\lambda_n}} - \mathrm{cap}(\Gamma) \approx C\,\frac{1}{n^s}$.

When ratio asymptotics were useless

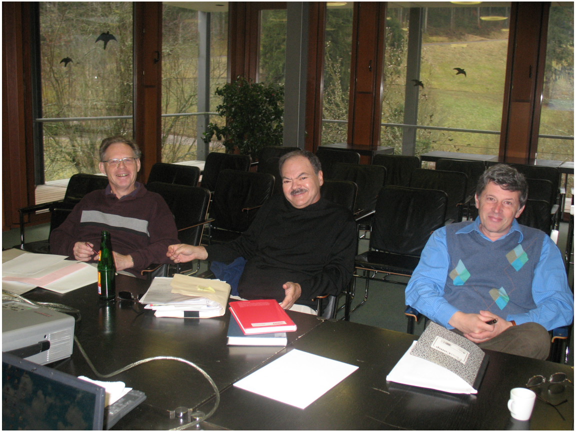
The conventional (Monomial) Gram-Schmidt process was, indeed, a true obstacle back then:

Henrici, Computational Complex Analysis, III (1986)

...However, the construction of a long sequence of orthogonal functions by means of the Gram-Schmidt process may run into difficulties, and the author knows of no nontrivial example where an accurate determination of $\Phi(z)$ *via* $\sqrt{\frac{n}{n+1}}\frac{p_n(z)}{p_{n-1}(z)}$ *was actually carried over.*

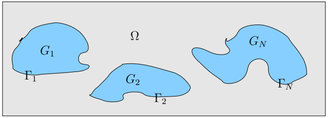
Discovery of an archipelago

Research in Pairs, Oberwolfach, January 2008



Bjorn Gustafsson, Ed Saff, Mihai Putinar

Asymptotics in an archipelago



Let $g_\Omega(z,\infty)$ denote the Green function of $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ with pole at ∞.

Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic. Then, for $n \in \mathbb{N}$:

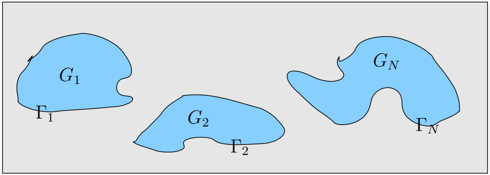
(i) *There exists a positive constant C, so that*

$$|p_n(z)| \leq \frac{C}{\operatorname{dist}(z,\Gamma)} \sqrt{n} \exp\{n\, g_\Omega(z,\infty)\}, \quad z \notin \overline{G}.$$

(ii) *For every $\epsilon > 0$ there exist a constant $C_\epsilon > 0$, such that*

$$|p_n(z)| \geq C_\epsilon \sqrt{n} \exp\{n\, g_\Omega(z,\infty)\}, \quad \operatorname{dist}(z, \mathrm{Co}(\overline{G})) \geq \epsilon.$$

Leading coefficients for an archipelago



Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic, $j = 1, 2, \ldots, N$. Then, for $n \in \mathbb{N}$,

$$c_1(\Gamma)\sqrt{\frac{n+1}{\pi}}\frac{1}{\operatorname{cap}(\Gamma)^{n+1}} \le \lambda_n \le c_2(\Gamma)\sqrt{\frac{n+1}{\pi}}\frac{1}{\operatorname{cap}(\Gamma)^{n+1}}.$$

This, in particular, shows that the Arnoldi GS for Bergman polynomials, in the archipelago case, is stable. However, in view of the strong asymptotics in the single island case, where $c_1(\Gamma) \equiv 1$ and $c_2(\Gamma) \equiv 1 + O(\rho^n)$, $0 < \rho < 1$, this looks a bit awkward...

Leading coefficients for a lemniscate

However, the following result shows that we can not expect to do any better, in general, than the previous double inequality.

Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Consider the lemniscate $G := \{z : |z^m - 1| < r^m\}$, $m \geq 2$, $0 < r < 1$, *and note that* $\text{cap}(\Gamma) = r$. *Then, the sequence*

$$\lambda_n \, \text{cap}(\Gamma)^{n+1} \sqrt{\frac{\pi}{n+1}}, \quad n \in \mathbb{N},$$

has exactly m limit points:

$$\frac{1}{r^{m-1}}, \frac{1}{r^{m-2}}, \ldots, \frac{1}{r}, 1.$$

A task regarding ratio asymptotics

The important class **Reg** of measures of orthogonality was introduced by Stahl and Totik in *General Orthogonal Polynomials*, CUP (1992). Recall that $\mu \in$ **Reg** if

$$\boxed{\lim_{n\to\infty} \lambda_n^{1/n}(\mu) = \frac{1}{\mathrm{cap}(\Gamma)}}.$$

Motivated by the crucial properties of the ratio asymptotics outlined above, we have proposed the following:

Problem

Give characterizations for measures μ, for which it holds that:

$$\boxed{\lim_{n\to\infty} \frac{\lambda_{n+1}(\mu)}{\lambda_n(\mu)} = \frac{1}{\mathrm{cap}(\Gamma)}}.$$

Recovery from area moments

Truncated Moments Problem

Given the finite $n+1 \times n+1$ section $[\mu_{m,k}]_{m,k=0}^{n}$,

$$\mu_{m,k} := \int_G z^m \overline{z}^k dA(z),$$

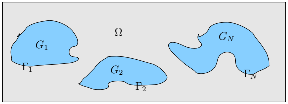
of the infinite complex moment matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$, associated with an archipelago G, compute a good approximation to its boundary Γ.

Theorem (Davis & Pollak, Trans. AMS, 1956)

The infinite matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$ *defines uniquely* Γ.

This leads to applications in 2D geometric tomography, through the Radon transform.

Discovery of an archipelago



Archipelago Recovery Algorithm
Gustafsson, Putinar, Saff & St, Adv. Math., 2009.

(I) Use the Arnoldi GS to compute $p_0, p_1, \ldots, p_n$, from $[\mu_{m,k}]_{m,k=0}^{n}$.

(II) Form the recovery function $\Lambda_n : \mathbb{C} \to (0, +\infty)$, with

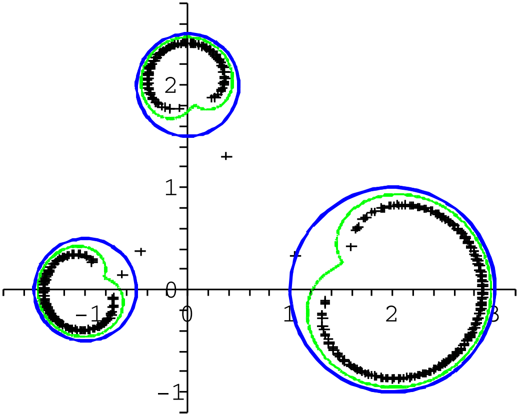
$$\Lambda_n(z) := \left[\sum_{k=0}^{n} |p_k(z)|^2 \right]^{-1/2}.$$

(III) Plot the zeros of p_j, $j = 1, 2, \ldots, n$.

(IV) Plot the level curves of the function $\Lambda_n(x + iy)$, on a suitable rectangular frame for (x, y) that surrounds the plotted zero set.

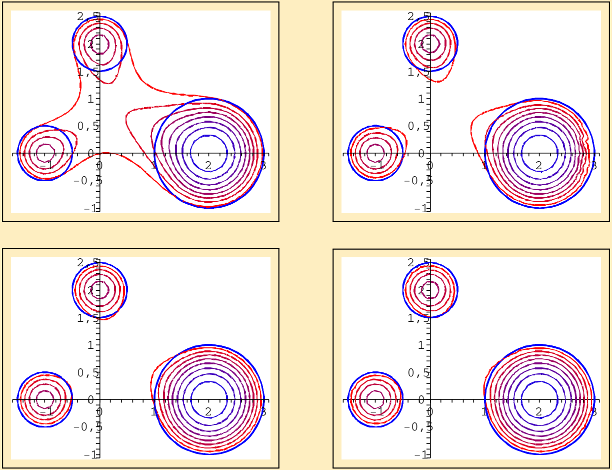
Three-disks

Zeros of the Bergman polynomials p_{140}, p_{150} and p_{160}.



Theory in: Gustafsson, Putinar, Saff & St, Adv. Math., 2009.

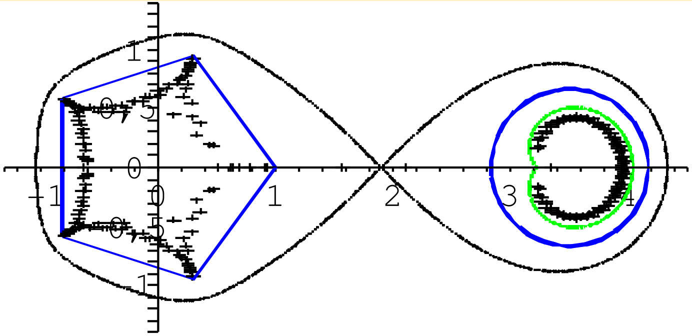
Recovery of three disks



Level lines of $\Lambda_n(x+iy)$ on $\{(x,y): -1 \leq x \leq 4, -2 \leq y \leq 2\}$, for $n = 25, 50, 75, 100$.

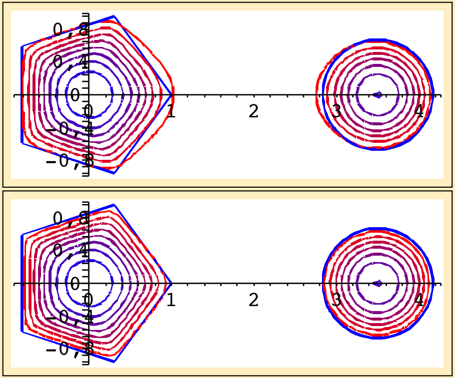
Pentagon and disk

Zeros of the Bergman polynomials p_{80}, p_{90} and p_{100}.



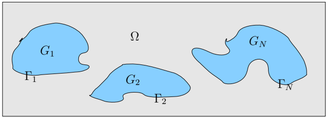
Theory in: Gustafsson, Putinar, Saff & St, Adv. Math., 2009.

Recovery of pentagon and disk



Level lines of $\Lambda_n(x+iy)$ on $\{(x,y) : -2 \leq x \leq 5, -2 \leq y \leq 2\}$, for $n = 25, 50$.

Why the recovery algorithm works?



Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic and let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$. Then, as $n \to \infty$:

$$
\begin{aligned}
\Lambda_n(z) &\asymp \operatorname{dist}(z, \Gamma), \quad z \in G. \\
\Lambda_n(z) &\asymp \frac{1}{n}, \quad z \in \Gamma. \\
\Lambda_n(z) &\asymp \frac{1}{\sqrt{n}} \exp\{-n\, g_\Omega(z, \infty)\}, \quad z \in \Omega.
\end{aligned}
$$

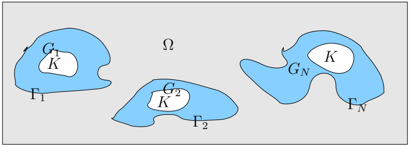
Note: The values of the Green function $g_\Omega(z, \infty)$ increase from 0 on Γ to $+\infty$ at ∞.

Archipelagoes with Lakes

Research in Pairs, Oberwolfach, January 2011

Ed Saff, Vilmos Totik, Herbert Stahl

Bergman polynomials on archipelago with lakes



With K is a compact subset of G, set $\boxed{G^* := G \setminus K}$ and consider

$$\langle f, g\rangle_{G^*} := \int_{G^*} f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G^*)} := \langle f, f\rangle_{G^*}^{1/2}.$$

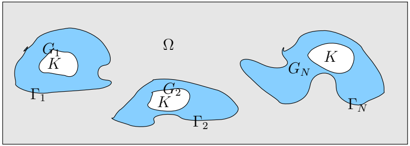
The Bergman polynomials $\{p_n^*\}_{n=0}^{\infty}$ of G^* are the unique orthonormal polynomials w.r.t. the area measure on G^*:

$$\langle p_m^*, p_n^*\rangle_{G^*} = \int_{G^*} p_m^*(z)\overline{p_n^*(z)}dA(z) = \delta_{m,n},$$

with

$$p_n^*(z) = \lambda_n^* z^n + \cdots, \quad \lambda_n^* > 0, \quad n = 0, 1, 2, \ldots.$$

Comparison between p_n and p_n^*, for Γ_j Jordan.



$$\Lambda_n^*(z) := \left[\sum_{k=0}^{\infty} |p_n^*(z)|^2\right]^{-1/2}$$

Theorem (Saff, Stahl, St & Totik)

Assume that every Γ_j is a Jordan curve. Then:

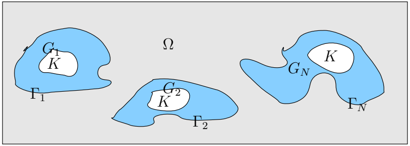
(i) $\lim_{n\to\infty} \lambda_n^*/\lambda_n = 1.$

(ii) $\lim_{n\to\infty} \|p_n^* - p_n\|_{L^2(G)} = 0.$

(iii) $\lim_{n\to\infty} p_n^*(z)/p_n(z) = 1,$ *locally uniformly in* $\overline{\mathbb{C}} \setminus \mathrm{Co}(G)$.

(iv) $\lim_{n\to\infty} \Lambda_n^*(z)/\Lambda_n(z) = 1,$ *locally uniformly in* $\overline{\mathbb{C}} \setminus \overline{G}$.

Comparison between p_n and p_n^*



Conclusion

The Bergman polynomials on an archipelago are "determined" by a strip near the outer boundaries.

This leads to an reconstruction algorithm for an archipelago having lakes.

Comparison between p_n and p_n^*, for Γ_j smooth.

Theorem (Saff, Stahl, St & Totik)

Assume that every boundary curve Γ_j is $C^{p+\alpha}$-smooth, with some $p \in \mathbb{N}$ and $0 < \alpha < 1$. Then,

(i) $\lambda_n^*/\lambda_n = 1 + O\left(\frac{1}{n^{2(p+\alpha-1)}}\right)$.

(ii) $\|p_n^* - p_n\|_{L^\infty(G)} = 1 + O\left(\frac{1}{n^{p+\alpha-2}}\right)$.

(iii) $p_n^*(z)/p_n(z) = 1 + O\left(\frac{1}{n^{p+\alpha-1}}\right)$, *locally uniformly in* $\overline{\mathbb{C}} \setminus \overline{G}$.

(iv) $\Lambda_n^*(z)/\Lambda_n(z) = 1 + O\left(\frac{1}{n^{2(p+2\alpha)-3}}\right)$, *locally uniformly in* $\overline{\mathbb{C}} \setminus \overline{G}$.

Furthermore, if every Γ_j is analytic, then the above hold with $O(\varrho^n)$, where $0 < \varrho < 1$.

In the above O depends on both the boundary Γ and the lake K.

Discovery of an archipelago having lakes

Archipelago-Lakes Recovery Algorithm: Phase A
Saff, Stahl, St & Totik

(I) Use the Arnoldi GS to compute $p_0^*, p_1^*, \dots, p_n^*$, from the given set of moments $\mu_{m,k}^*$ of G^*.

(II) Form the recovery function

$$\Lambda_n^*(z) := \left[\sum_{k=0}^{n} |p_k^*(z)|^2\right]^{-1/2}.$$

(III) Plot the zeros of p_j^*, $j = 1, 2, \dots, n$.

(IV) Plot the level curves of the function $\Lambda_n^*(x+iy)$, on a suitable rectangular frame for (x, y) that surrounds the plotted zero set.

This should produce a good approximation $\widetilde{G}$ of G.

Discovery of an archipelago having lakes

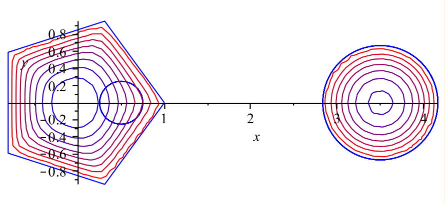
Archipelago-Lakes Recovery Algorithm: Phase B
Saff, Stahl, St & Totik

(I) Use $\widetilde{G}$ to calculate the approximate moments $\widetilde{\mu}_{m,k}$ and set $\widehat{\mu}_{m,k} := \widetilde{\mu}_{m,k} - \mu^*_{m,k}$.

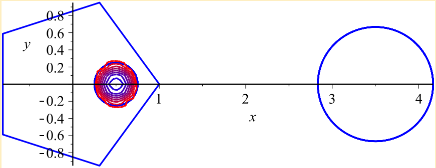
(II) Perform steps (I)–(IV) of Phase A with $\widehat{\mu}_{m,k}$ as input, in the place of $\mu^*_{m,k}$.

This will produce an approximation $\widetilde{K}$ of the lake K.
Beware though: Rivers (slits) cannot be recovered using area moments ...

Recovery of pentagon G_1 (with a disk lake K) and disk G_2



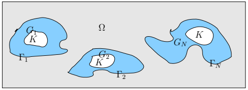
Phase A



Phase B

Level lines of $\Lambda_n(x+iy)$ on $\{(x,y) : -2 \le x \le 5, -2 \le y \le 2\}$, for $n = 80$.

Why this recovery algorithm works?



Theorem

Assume that every Γ_j is Jordan curve. Then, as $n \to \infty$:

$$\begin{aligned} \Lambda_n^*(z) &\geq \sqrt{\pi}\,\mathrm{dist}(z,\Gamma), \quad z \in G. \\ \Lambda_n^*(z) &\leq A(G)\exp\left(n\{\varepsilon_n(z) - g_\Omega(z,\infty)\}\right), \quad z \in \Omega, \end{aligned}$$

where

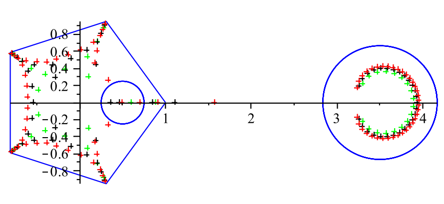
$$\varepsilon_n(z) \geq 0 \quad \text{and} \quad \lim_{n\to\infty} \varepsilon_n(z) = 0,$$

locally uniformly in Ω.

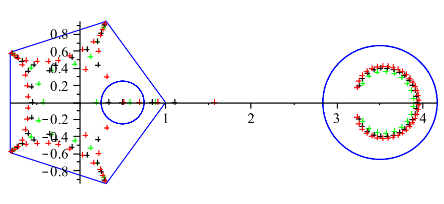
Reminder: The values of the Green function $g_\Omega(z,\infty)$ increase from 0 on Γ to $+\infty$ at ∞.

A very suggestive example...

Zeros of p_{40}, p_{60} and p_{80}, for pentagon G_1, disk lake K, disk G_2



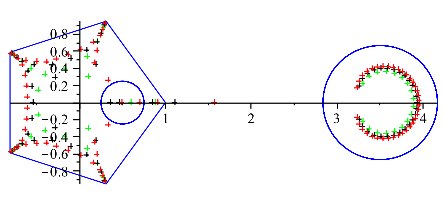
$$d\mu = dA|_{G_1} + dA|_{G_2} - dA|_K$$



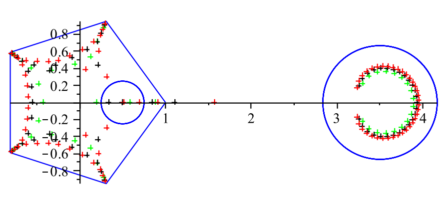
$$d\mu = dA|_{G_1} + dA|_{G_2} + dA|_K$$

A very suggestive example...

Zeros of p_{40}, p_{60} and p_{80}, for pentagon G_1, disk lake K, disk G_2



$$d\mu = dA|_{G_1} + dA|_{G_2} - dA|_K$$



$$d\mu = dA|_{G_1} + dA|_{G_2} + dA|_K$$

Theory for general measures

Let μ_1 and μ_2 be finite Borel measures with compact and infinite supports $S_1 := \operatorname{supp}(\mu_1)$ and $S_2 := \operatorname{supp}(\mu_2)$ in $\mathbb{C}$ and set

$$\boxed{\mu := \mu_1 + \mu_2}.$$

Let $\{p_n(\mu, z)\}_{n=0}^{\infty}$ and $\{p_n(\mu_1, z)\}_{n=0}^{\infty}$ denote the two sequences of orthonormal polynomials defined, respectively, by the inner products

$$\langle f, g\rangle_\mu := \int f(z)\overline{g(z)}d\mu(z) \quad \text{and} \quad \langle f, g\rangle_{\mu_1} := \int f(z)\overline{g(z)}d\mu_1(z),$$

having positive leading coefficients $\{\lambda_n(\mu)\}_{n=0}^{\infty}$ and $\{\lambda_n(\mu_1)\}_{n=0}^{\infty}$.

Comparison between $p_n(\mu, z)$ and $p_n(\mu_1, z)$.

Theorem

Assume that

$$\boxed{\lim_{n\to\infty} \|p_n(\mu_1, \cdot)\|_{L^2(\mu_2)} = 0.}$$

Then the following hold:

(i) $\lim_{n\to\infty} \gamma_n(\mu_1)/\gamma_n(\mu) = 1.$

(ii) $\lim_{n\to\infty} \|p_n(\mu_1, \cdot) - p_n(\mu, \cdot)\|_{L^2(\mu)} = 0.$

(iii) $\lim_{n\to\infty} p_n(\mu_1, z)/p_n(\mu, z) = 1,$
uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\mu))$.

(iv) *If, in addition,* $\mathrm{cap}(S_1) > 0$ *and* $\lim_{n\to\infty} \max_{z\in S_2} \sum_{j=n}^{\infty} |p_j(\mu_1, z)|^2 = 0$, *then*

$$\lim_{n\to\infty} \Lambda_n(\mu_1, z)/\Lambda_n(\mu, z) = 1,$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{supp}(\mu)$.

Annular region $G^* := \mathbb{D} \setminus \overline{D(a,r)}$

Zeros of p_{80}, p_{90} and p_{100}, for $r = 0.25$ and $a = 0.2, 0.4, 0.5, 0.7$.