Bergman Polynomials on Islands and Archipelaga: Construction, Asymptotics, Zeros and Shape Recovery

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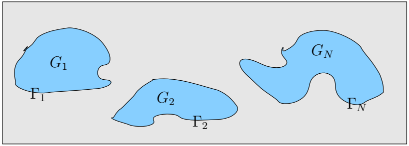
Bergman polynomials on an archipelago



Γ_j, $j = 1, \ldots, N$, a system of disjoint and mutually exterior Jordan curves in $\mathbb{C}$, $\boxed{G_j := \text{int}(\Gamma_j)}$, $\boxed{\Gamma := \cup_{j=1}^N \Gamma_j}$, $\boxed{G := \cup_{j=1}^N G_j}$.

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)} dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}$$

The Bergman polynomials $\{p_n\}_{n=0}^\infty$ of G are the unique orthonormal polynomials w.r.t. the area measure on G:

$$\langle p_m, p_n \rangle = \int_G p_m(z)\overline{p_n(z)} dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

Construction of the Bergman polynomials

Algorithm: Monomial Gram-Schmidt (GS)

Apply the Gram-Schmidt process to the monomials

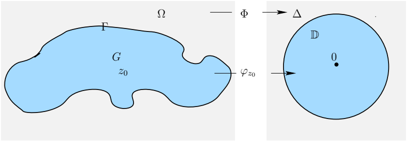
$$1, z, z^2, z^3, \ldots$$

Main ingredient: the moments

$$\mu_{m,k} := \langle z^m, z^k \rangle = \int_G z^m \overline{z}^k dA(z), \quad m, k = 0, 1, \ldots.$$

The above algorithm has been been suggested, and used in practise, by pioneers of Numerical Conformal Mapping, like P. Davis and D. Gaier, in the 1960's, as the standard procedure for constructing Bergman polynomials. It was still treated as the standard method by leaders of the subject, like P. Henrici (*Computational Complex Analysis*, Vols I,II and III), in the 1980's.

The interior conformal mapping for a single island



In the single island case $N = 1$, we fix $z_0 \in G$ and consider the interior conformal mapping:

$$\varphi_{z_0} : G \to \mathbb{D} \quad \text{with} \quad \varphi_{z_0}(z_0) = 0, \quad \text{and} \quad \varphi'_{z0}(z_0) > 0.$$

Computationally, it is easier to approximate the normalized mapping

$$f_0(z) := \frac{\varphi_{z0}(z)}{\varphi'_{z0}(z_0)}, \quad \text{so that} \quad f_0(z_0) = 0 \quad \text{and} \quad f'_0(z_0) = 1.$$

Series representation for the Bergman kernel

Let $K(z, z_0)$ denote the reproducing kernel of the Bergman space $L^2_a(G)$ w.r.t. the point evaluation at z_0. Then,

$$K(z, z_0) = \sum_{j=0}^{\infty} \overline{p_j(z_0)} p_j(z), \quad \text{locally uniformly in } G.$$

The Bergman kernel $K(\cdot, z_0)$ is related to the mapping function f_0 by

$$f_0(z) = \frac{1}{K(z_0, z_0)} \int_{z_0}^{z} K(\zeta, z_0) d\zeta. \tag{1}$$

This relation yields the Bergman kernel method: Construct the Bergman polynomials $\{p_n\}_{n=0}^{n}$ of G and form the finite Fourier sum

$$K_n(z, z_0) = \sum_{j=0}^{n} \overline{p_j(z_0)} p_j(z).$$

Then, replace $K(\zeta, z_0)$ by $K_n(\zeta, z_0)$ in (1) to approximate $f_0(z)$.

Instability Indicator

The GS method is notorious for its instability. For measuring it, when orthonormalizing a system $S_n := \{u_0, u_1, \ldots, u_n\}$ of functions, in a Hilbert space with norm $\|\cdot\|$, the following instability indicator has been proposed by J.M. Taylor, (Proc. R.S. Edin., 1978):

$$\boxed{I_n := \frac{\|u_n\|^2}{\min_{u\in \mathrm{span}(S_{n-1})} \|u_n - u\|^2}, \quad n \in \mathbb{N}.}$$

Note that, when S_n is an orthonormal system, then $\boxed{I_n = 1}$. When S_n is linearly dependent then $\boxed{I_n = \infty}$. Also, if $G_n := [\langle u_m, u_k\rangle]_{m,k=0}^{n}$, denotes the Gram matrix associated with S_n then,

$$\boxed{\kappa(G_n) \geq I_n},$$

where $\kappa(G_n) := \|G_n\|\|G_n^{-1}\|$ is the spectral condition number of G_n.

Instability of the Monomial GS process: Island

In the single-component case $N = 1$, consider the set of monomials $S_n = \{1, z, z^2, \ldots, z^n\}$. Then, for the resulting GS process we have the following result:

Theorem (Papamichael & Warby, Numer. Math., 1986)

Assume that the curve Γ *is piecewise-analytic without cusps and let*

$$\boxed{L := \|z\|_{L^\infty(\Gamma)}/\mathrm{cap}(\Gamma)} \quad (\geq 1),$$

where $\mathrm{cap}(\Gamma)$ *denotes the logarithmic capacity of* Γ. *Then,*

$$\boxed{c_1(\Gamma)\, L^{2n} \leq I_n \leq \; c_2(\Gamma)\, L^{2n}}.$$

Note that $L = 1$, iff $G \equiv \mathbb{D}_r$ and that I_n is sensitive to the relative position of G w.r.t. the origin. When G is the 8×2 rectangle centered at the origin, then $L = 3/\sqrt{2} \approx 2.12$. In this case, $I_{25} \asymp 10^{16}$ and the method breaks down in MATLAB or FORTRAN, for $n = 25$.

Instability of the Monomial GS process: Archipelago

As it is expected, things cannot get any better in the archipelago case.

Theorem

Let G be an archipelago and consider the application of the GS process to the monomials $S_n = \{1, z, z^2, \ldots, z^n\}$. Assume that boundary Γ of G satisfies an interior cone condition at the point z_0, where $|z_0| = \|z\|_{L^\infty(\Gamma)}$ and let

$$\boxed{L := \|z\|_{L^\infty(\Gamma)}/\mathrm{cap}(\Gamma)} \quad (\geq 1).$$

Then

$$\boxed{c_1(\Gamma)\, L^{2n} \leq I_n \leq c_2(\Gamma)\, L^{2n}}.$$

The Arnoldi algorithm for OP's

Let μ be a (non-trivial) finite Borel measure with compact support $\text{supp}(\mu)$ on $\mathbb{C}$ and consider the associated series of orthonormal polynomials

$$p_n(z,\mu) := \lambda_n(\mu) z^n + \cdots, \quad \lambda_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

generated by the inner product

$$\langle f, g\rangle_\mu = \int f(z)\overline{g(z)}d\mu(z), \quad \|f\|_{L^2(\mu)} := \langle f, g\rangle_\mu^{1/2}.$$

Arnoldi GS for Orthonormal Polynomials

At the n-th step, apply GS to orthonormalize the polynomial zp_{n-1} (instead of z^n) against the (already computed) orthonormal polynomials $\{p_0, p_1, \ldots, p_{n-1}\}$.

Used by Gragg & Reichel, in Linear Algebra Appl. (1987), for the construction of Szegö polynomials.

Stability of the Arnoldi GS

In the case of the Arnoldi GS, the instability indicator is given by:

$$I_n = \frac{\|zp_{n-1}\|^2_{L^2(\mu)}}{\min_{p\in\mathbb{P}_{n-1}} \|zp_{n-1} - p\|^2_{L^2(\mu)}}, \quad n \in \mathbb{N}.$$

Theorem (St, arXiv, 2012)

It holds,

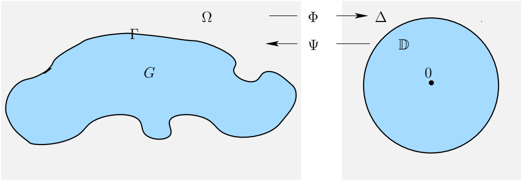
$$1 \leq I_n \leq \|z\|_{L^\infty(\text{supp}(\mu))} \frac{\lambda_n^2(\mu)}{\lambda_{n-1}^2(\mu)}, \quad n \in \mathbb{N}.$$

It turns out that when $d\mu \equiv |dz|$ (Szegö polynomials), or $d\mu \equiv dA$ (Bergman polynomials), then

$$\boxed{c_1(\mu) \leq \frac{\lambda_n(\mu)}{\lambda_{n-1}(\mu)} \leq c_2(\mu)}, \quad n \in \mathbb{N}.$$

When $d\mu \equiv w(x)dx$ on $[a, b] \subset \mathbb{R}$, this ratio tends to a constant.

Asymptotics: Single-component case $N = 1$



$$\Omega := \overline{\mathbb{C}} \setminus \overline{G}$$

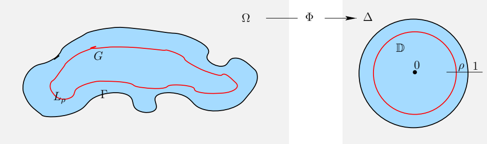
$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots . \qquad \text{cap}(\Gamma) = 1/\gamma$$

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots . \qquad \text{cap}(\Gamma) = b$$

The Bergman polynomials of G:

$$p_n(z) = \lambda_n z^n + \cdots , \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots .$$

Strong asymptotics when Γ is analytic



T. Carleman, Ark. Mat. Astr. Fys. (1922)

If $\rho < 1$ is the smallest index for which Φ is conformal in $\mathrm{ext}(L_\rho)$, then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\rho^{2n},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}}, \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \le c_2(\Gamma)\sqrt{n}\,\rho^n, \quad z \in \overline{\Omega}.$$

Strong asymptotics when Γ is smooth

We say that $\Gamma \in C(p, \alpha)$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$, if Γ is given by $z = g(s)$, where s is the arclength, with $g^{(p)} \in \text{Lip}\alpha$. Then both Φ and $\Psi := \Phi^{-1}$ are p times continuously differentiable in $\overline{\Omega} \setminus \{\infty\}$ and $\overline{\Delta} \setminus \{\infty\}$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \text{Lip}\alpha$.

P.K. Suetin, Proc. Steklov Inst. Math. AMS (1974)

Assume that $\Gamma \in C(p+1, \alpha)$, with $p + \alpha > 1/2$. Then, for $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,} \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\frac{1}{n^{2(p+\alpha)}},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}},$$

where

$$|A_n(z)| \le c_2(\Gamma)\,\frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}.$$

Strong asymptotics for Γ non-smooth

Theorem (St, C. R. Acad. Sci. Paris, 2010)

Assume that Γ *is piecewise analytic without cusps. Then, for* $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \textit{where} \quad 0 \le \alpha_n \le c(\Gamma)\,\frac{1}{n},$$

and for any $z \in \Omega$,

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + A_n(z)\right\}},$$

where

$$|A_n(z)| \le \frac{c_1(\Gamma)}{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

Letting $z \to \infty$, we recover the result for the leading coefficients.

Strong asymptotics for Γ non-smooth

The following example shows that we cannot expect the strong asymptotics for p_n to hold at corner points of Γ.

Example

Assume that Γ is the square with corners at 1, i, -1 and $-i$ and let z_0 denote any corner of Γ. Then, the following expansion is valid near z_0:

$$\Phi(z) = \Phi(z_0) + a_1(z - z_0)^{2/3} + a_2(z - z_0)^{4/3} + \cdots ,$$

with $a_1 \neq 0$. Hence, near z_0:

$$\Phi'(z) = \frac{2}{3} a_1(z - z_0)^{-1/3} + \frac{4}{3} a_2(z - z_0)^{1/3} + \cdots ,$$

which leads to $\boxed{\Phi'(z_0) = \infty}$ and makes impossible the formula

$$p_n(z_0) = \sqrt{\frac{n+1}{\pi}} \Phi^n(z_0)\Phi'(z_0)\{1 + A_n(z_0)\}, \quad A_n(z_0) = o(1).$$

Ratio asymptotics for λ_n

Corollary (St, C. R. Acad. Sci. Paris, 2010)

$$\boxed{\sqrt{\frac{n+1}{n}}\,\frac{\lambda_{n-1}}{\lambda_n} = \mathrm{cap}(\Gamma) + \xi_n}, \quad \textit{where} \quad |\xi_n| \leq c(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}.$$

The above relation provides the means for computing approximations to the capacity of Γ, by using only the leading coefficients of the Bergman polynomials. In addition, it implies for the Instability Indicator I_n of the Arnoldi GS:

Corollary

$$\boxed{c_1(\Gamma) \leq I_n \leq c_2(\Gamma)}, \quad n \in \mathbb{N}.$$

Hence, the Arnoldi GS for Bergman polynomials, in the single component case, is stable.

Ratio asymptotics for $p_n(z)$

Corollary (St, C. R. Acad. Sci. Paris, 2010)

For any $z \in \Omega$, and sufficiently large $n \in \mathbb{N}$,

$$\boxed{\sqrt{\frac{n}{n+1}}\,\frac{p_n(z)}{p_{n-1}(z)} = \Phi(z)\{1 + B_n(z)\}},$$

where

$$|B_n(z)| \leq \frac{c_1(\Gamma)}{\sqrt{\mathrm{dist}(z,\Gamma)\,|\Phi'(z)|}}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

The above relation, combined with the Arnoldi GS for constructing the Bergman polynomials $p_n(z)$, $n = 0, 1, \ldots$, provides an efficient method for computing approximations to $\Phi(z)$.

Example: Computing the capacity of a Square

n	$\sqrt{\frac{n+1}{n}\frac{\lambda_{n-1}}{\lambda_n}}$	$\sqrt{\frac{n+1}{n}\frac{\lambda_{n-1}}{\lambda_n}} - \text{cap}(\Gamma)$	s
100	0.834 640 612	1.37e-05	-
110	0.834 638 233	1.14e-05	1.9902
120	0.834 636 420	9.58e-06	1.9911
130	0.834 635 009	8.16e-06	1.9918
140	0.834 633 888	7.04e-06	1.9924
150	0.834 632 982	6.14e-06	1.9930
160	0.834 632 341	5.39e-06	1.9934
170	0.834 631 626	4.78e-06	1.9938
180	0.834 631 111	4.26e-06	1.9942
190	0.834 630 674	3.83e-06	1.9945
200	0.834 630 301	3.46e-06	1.9949

Γ: Square with corners at $1, i, -1, -i$. $\boxed{\text{cap}(\Gamma) = 0.834\,626\,841\cdots.}$

s: tests the hypothesis $\sqrt{\frac{n+1}{n}\frac{\lambda_{n-1}}{\lambda_n}} - \text{cap}(\Gamma) \approx C\,\frac{1}{n^s}$.

When ratio asymptotics were useless

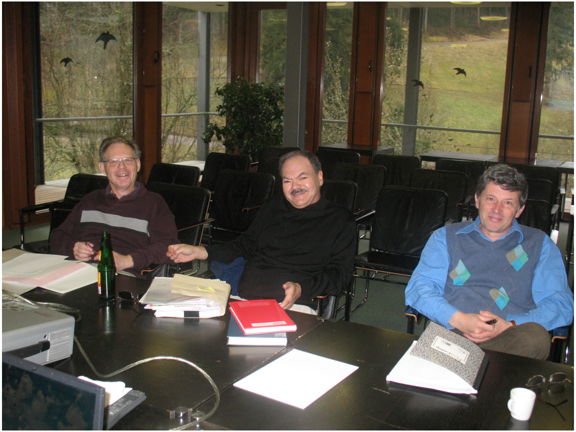
The conventional (Monomial) Gram-Schmidt process was, indeed, a true obstacle back then:

Henrici, Computational Complex Analysis, III (1986)

...However, the construction of a long sequence of orthogonal functions by means of the Gram-Schmidt process may run into difficulties, and the author knows of no nontrivial example where an accurate determination of $\Phi(z)$ *via* $\sqrt{\frac{n}{n+1}}\frac{p_n(z)}{p_{n-1}(z)}$ *was actually carried over.*

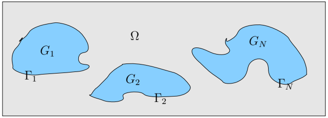
Discovery of an archipelago

Research in Pairs, Oberwolfach, January 2008



Bjorn Gustafsson, Ed Saff, Mihai Putinar

Asymptotics in an archipelago



Let $g_\Omega(z,\infty)$ denote the Green function of $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ with pole at ∞.

Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic. Then, for $n \in \mathbb{N}$:

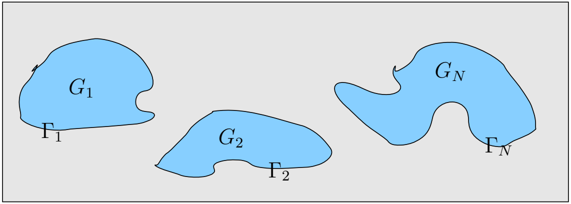
(i) *There exists a positive constant C, so that*

$$|p_n(z)| \leq \frac{C}{\mathrm{dist}(z,\Gamma)}\sqrt{n}\exp\{n\, g_\Omega(z,\infty)\}, \quad z \notin \overline{G}.$$

(ii) *For every $\epsilon > 0$ there exist a constant $C_\epsilon > 0$, such that*

$$|p_n(z)| \geq C_\epsilon \sqrt{n}\exp\{n\, g_\Omega(z,\infty)\}, \quad \mathrm{dist}(z, \mathrm{Co}(\overline{G})) \geq \epsilon.$$

Leading coefficients for an archipelago



Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic, $j = 1, 2, \ldots, N$. Then, for $n \in \mathbb{N}$,

$$c_1(\Gamma)\sqrt{\frac{n+1}{\pi}}\frac{1}{\operatorname{cap}(\Gamma)^{n+1}} \leq \lambda_n \leq c_2(\Gamma)\sqrt{\frac{n+1}{\pi}}\frac{1}{\operatorname{cap}(\Gamma)^{n+1}}.$$

In view of the strong asymptotics in the single island case, this looks a bit disappointing...

Leading coefficients for a lemniscate

However, the following result shows that we can not expect to do any better, in general, than the previous double inequality.

Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Consider the lemniscate $G := \{z : |z^m - 1| < r^m\}$, $m \geq 2$, $0 < r < 1$, *and note that* $\operatorname{cap}(\Gamma) = r$. *Then, the sequence*

$$\lambda_n \operatorname{cap}(\Gamma)^{n+1} \sqrt{\frac{\pi}{n+1}}, \quad n \in \mathbb{N},$$

has exactly m limit points:

$$\frac{1}{r^{m-1}}, \frac{1}{r^{m-2}}, \ldots, \frac{1}{r}, 1.$$

A question regarding ratio asymptotics

The important class **Reg** of measures of orthogonality was introduced by Stahl and Totik in *General Ortho Polys*, CUP (1992). Recall that $\mu \in$ **Reg** if

$$\boxed{\lim_{n\to\infty} \lambda_n^{1/n}(\mu) = \frac{1}{\mathrm{cap}(\Gamma)}}.$$

Motivated by the crucial properties of the ratio asymptotics outlined above, we have asked the following:

Question

Characterize all the measures of orthogonality μ, for which it holds:

$$\boxed{\lim_{n\to\infty} \frac{\lambda_{n+1}(\mu)}{\lambda_n(\mu)} = \frac{1}{\mathrm{cap}(\Gamma)}} \quad \text{or} \quad \boxed{c_1 \le \frac{\lambda_{n+1}(\mu)}{\lambda_n(\mu)} \le c_2}.$$

Clearly, such measures will be in the **Reg** class.

Recovery from area moments

Truncated Moments Problem

Given the finite $n+1 \times n+1$ section $[\mu_{m,k}]_{m,k=0}^{n}$,

$$\mu_{m,k} := \int_G z^m \overline{z}^k \, dA(z),$$

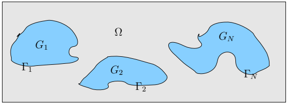
of the infinite complex moment matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$, associated with an archipelago G, compute a good approximation to its boundary Γ.

Theorem (Davis & Pollak, Trans. AMS, 1956)

The infinite matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$ defines uniquely Γ.

This leads to applications in 2D geometric tomography, through the Radon transform.

Discovery of an archipelago



Archipelago Recovery Algorithm
Gustafsson, Putinar, Saff & St, Adv. Math., 2009.

(I) Use the Arnoldi GS to compute $p_0, p_1, \ldots, p_n$, from $[\mu_{m,k}]_{m,k=0}^n$.

(II) Form the square root of the Christoffel function

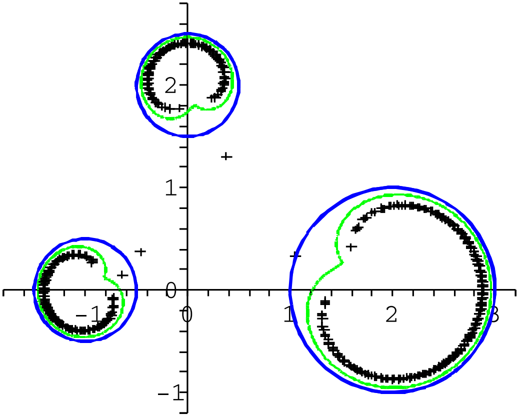
$$\Lambda_n(z) := 1 \Big/ \sqrt{\sum_{k=0}^{n} |p_k(z)|^2}.$$

(III) Plot the zeros of p_j, $j = 1, 2, \ldots, n$.

(IV) Plot the level curves of the function $\Lambda_n(x + iy)$, on a suitable rectangular frame for (x, y) that surrounds the plotted zero set.

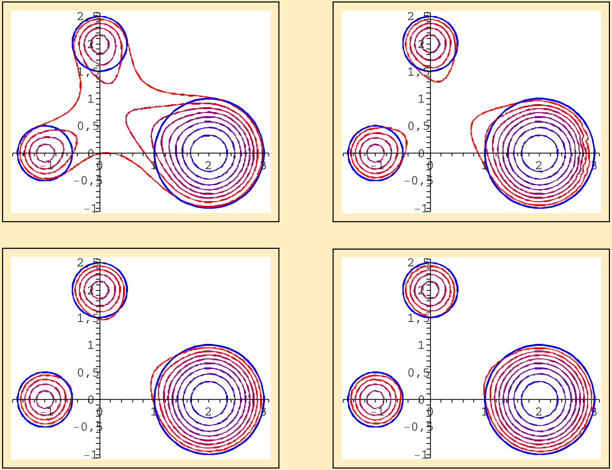
Three-disks

Zeros of the Bergman polynomials p_{140}, p_{150} and p_{160}.



Theory in: Gustafsson, Putinar, Saff & St, Adv. Math., 2009.

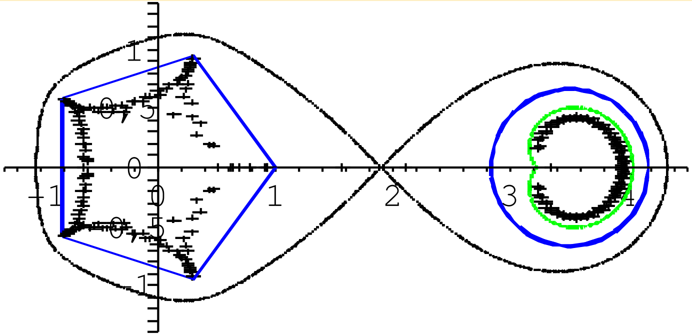
Recovery of three disks



Level lines of $\Lambda_n(x+iy)$ on $\{(x,y) : -1 \leq x \leq 4, -2 \leq y \leq 2\}$, for $n = 25, 50, 75, 100$.

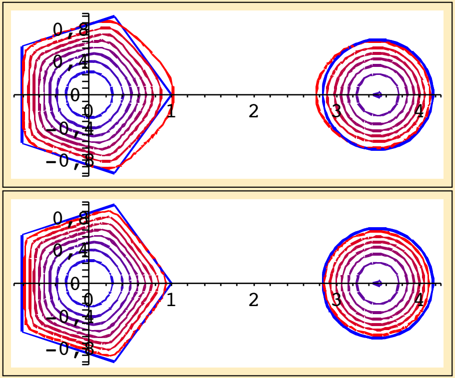
Pentagon and disk

Zeros of the Bergman polynomials p_{80}, p_{90} and p_{100}.



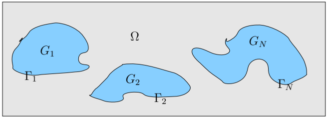
Theory in: Gustafsson, Putinar, Saff & St, Adv. Math., 2009.

Recovery of pentagon and disk



Level lines of $\Lambda_n(x+iy)$ on $\{(x,y) : -2 \le x \le 5, -2 \le y \le 2\}$, for $n = 25, 50$.

Theoretical support of the recovery algorithm



Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic and let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$. Then,

$$\begin{aligned} \Lambda_n(z) &\asymp \operatorname{dist}(z,\Gamma), \quad z \in G, \quad n \to \infty \\ \Lambda_n(z) &\asymp \frac{1}{n}, \quad z \in \Gamma, \quad n \to \infty \\ \Lambda_n(z) &\asymp \frac{1}{\sqrt{n}} \exp\{-n\, g_\Omega(z,\infty)\}, \quad z \in \Omega, \quad n \to \infty. \end{aligned}$$

where $g_\Omega(z,\infty)$ denotes the Green function of Ω with pole at infinity.

Note: $\boxed{g_\Omega(z,\infty) > 0, \quad z \in \Omega}$.

The basic tools for the distribution of zeros

- $K(z,\zeta)$: the Bergman (reproducing) kernel function of $L_a^2(G)$.
- $L_R := \{z : g_\Omega(z,\infty) = \log R\}$ the level lines of the Green function.
- $\varrho(\zeta) := \sup\{R : K(z,\zeta)$ *has an analytic continuation inside* $L_R\}$.
- The subharmonic function

$$h(z) := \begin{cases} g_\Omega(z,\infty), & z \in \overline{\Omega}, \\ -\log \varrho(z), & z \in G, \end{cases}$$

- The canonical measure $\boxed{\beta := \frac{1}{2\pi}\Delta h}$,
- ν_{p_n}: the normalized counting measure of zeros of p_n.
- $\mathcal{C}$: the set of weak-star cluster points of the sequence $\{\nu_{p_n}\}_{n=1}^{\infty}$, i.e., the set of measures σ for which there exists a subsequence $\mathcal{N}_\sigma \subset \mathbb{N}$ such that $\nu_{p_n} \xrightarrow{*} \sigma$, as $n \to \infty$, $n \in \mathcal{N}_\sigma$.
- μ_Γ: the equilibrium measure on the boundary Γ.

A basic result for the distribution of zeros

Theorem (Gustafsson, Putinar, Saff & St, Advances in Math, 2009)

(i) β *is a positive unit measure with support contained in* $\overline{G}$.

(ii) *The balayage of* β *onto* Γ *gives the equilibrium measure* μ_Γ*:*

$$\begin{cases} U^{\mu_\Gamma} \leq U^\beta \text{ in } \mathbb{C}, \\ U^\beta = U^{\mu_\Gamma} \text{ in } \Omega, \end{cases} \qquad \text{where} \quad U^\mu(z) = \int \log \frac{1}{|z-t|} d\mu(t).$$

(iii) $\mathcal{C}$ *is nonempty, and for any* $\sigma \in \mathcal{C}$,

$$\begin{cases} U^\beta \leq U^\sigma & \text{in } \mathbb{C}, \\ U^\sigma = U^\beta & \text{in the unbounded component of } \overline{\mathbb{C}} \setminus \operatorname{supp}\beta. \end{cases}$$

(iv) *The measure* β *is the lower envelope of* $\mathcal{C}$*:* $U^\beta = \operatorname{lsc}(\inf_{\sigma\in\mathcal{C}} U^\sigma)$.

(v) *If* $\mathcal{C}$ *has only one element, then this is* β *and*

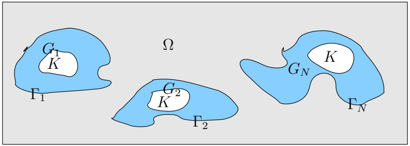
$$\nu_{p_n} \xrightarrow{*} \beta, \quad n \to \infty, \ n \in \mathbb{N}.$$

Archipelagoes with Lakes

Research in Pairs, Oberwolfach, January 2011

Ed Saff, Vilmos Totik, Herbert Stahl

Bergman polynomials on archipelago with lakes



With K is a compact subset of G, set $\boxed{G^* := G \setminus K}$ and consider

$$\langle f, g \rangle_{G^*} := \int_{G^*} f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G^*)} := \langle f, f \rangle_{G^*}^{1/2}.$$

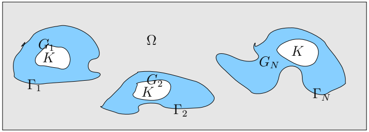
The Bergman polynomials $\{p_n^*\}_{n=0}^{\infty}$ of G^* are the unique orthonormal polynomials w.r.t. the area measure on G^*:

$$\langle p_m^*, p_n^* \rangle_{G^*} = \int_{G^*} p_m^*(z)\overline{p_n^*(z)}dA(z) = \delta_{m,n},$$

with

$$p_n^*(z) = \lambda_n^* z^n + \cdots, \quad \lambda_n^* > 0, \quad n = 0, 1, 2, \ldots.$$

Comparison between p_n and p_n^*, for Γ_j Jordan



$$\Lambda_n^*(z) := 1 \Big/ \sqrt{\sum_{k=0}^{\infty} |p_n^*(z)|^2}$$

Theorem (Saff, Stahl, St & Totik)

Assume that any boundary curve Γ_j in the archipelago is Jordan. Then, as $n \to \infty$ we have:

(i) $\lambda_n^*/\lambda_n \to 1$;

(ii) $\|p_n^* - p_n\|_{L^2(G)} \to 0$;

(iii) $\Lambda_n^*(z)/\Lambda_n(z) \to 1$, *locally uniformly in* $\overline{\mathbb{C}} \setminus \overline{G}$;

(iv) $p_n^*(z)/p_n(z) \to 1$, *locally uniformly in* $\overline{\mathbb{C}} \setminus \mathrm{Co}(G)$.

Comparison between p_n and p_n^*, for Γ_j smooth

Theorem (Saff, Stahl, St & Totik)

Assume that every boundary curve Γ_j is $C^{p+\alpha}$-smooth, with some $p \in \mathbb{N}$ and $0 < \alpha < 1$. Then,

(i) $\lambda_n^*/\lambda_n = 1 + O\left(\frac{1}{n^{2(p+\alpha)-2}}\right)$;

(ii) $\|p_n^* - p_n\|_{L^\infty(G)} = 1 + O\left(\frac{1}{n^{p+\alpha-2}}\right)$;

(iii) $\Lambda_n^*(z)/\Lambda_n(z) = 1 + O\left(\frac{1}{n^{2(p+\alpha)-3}}\right)$, *locally uniformly in* $\overline{\mathbb{C}} \setminus \overline{G}$;

(iv) $p_n^*(z)/p_n(z) = 1 + O\left(\frac{1}{n^{p+\alpha-1}}\right)$, *locally uniformly in* $\overline{\mathbb{C}} \setminus \overline{G}$.

Comparison between p_n and p_n^*

Conclusion

The Bergman polynomials on an archipelago are actually "determined" by a strip near the outer boundaries.

This property of Bergman polynomials leads to an reconstruction algorithm for archipelaga having lakes.

Discovery of an archipelago having lakes

Archipelago with Lakes Reconstruction Algorithm
Saff, Stahl, St & Totik

(I) Use the Arnoldi GS to compute $p_0^*, p_1^*, \dots, p_n^*$, from the given set of moments $\mu_{m,k}^*$ of G^*.

(II) Perform steps (2)–(4) of the Archipelago Reconstruction Algorithm with p_n^* in the place of p_n. Since $p_n^* \approx p_n$, this will produce an approximation of G.

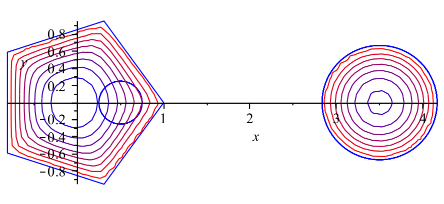
(III) Use this approximation of G to calculate the moments $\mu_{m,k}$.

(IV) Set $\widehat{\mu}_{m,k} := \mu_{m,k} - \mu_{m,k}^*$.

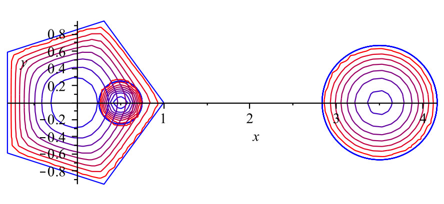
(V) Perform steps (1)–(4) of the Archipelago Reconstruction Algorithm, with $\widehat{\mu}_{m,k}$ in the place of $\mu_{m,k}$. This will produce an approximation to K.

(VI) Plot K against G.

Recovery of pentagon G_1 (with a disk lake K) and disk G_2



$$\mu = \mu^{G_1} + \mu^{G_2} - \mu^K$$

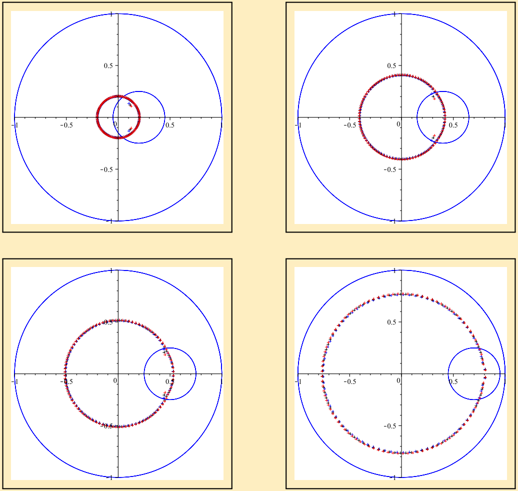


$$\mu^K = \mu^{G_1} + \mu^{G_2} - \mu$$

Level lines of $\Lambda_n(x + iy)$ on $\{(x, y) : -2 \le x \le 5, -2 \le y \le 2\}$, for $n = 80$.

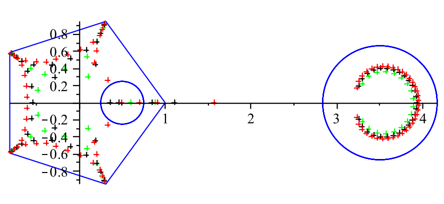
Annular region $G^* := \mathbb{D} \setminus \overline{D(a,r)}$

Zeros of p_{80}, p_{90} and p_{100}, for $r = 0.25$ and $a = 0.2, 0.4, 0.5, 0.7$.

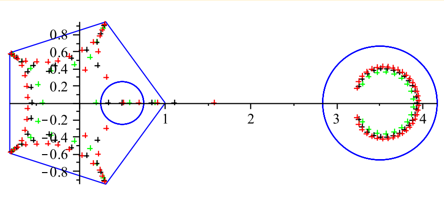


A very suggestive example...

Zeros of p_{40}, p_{60} and p_{80}, for pentagon G_1, disk lake K, disk G_2



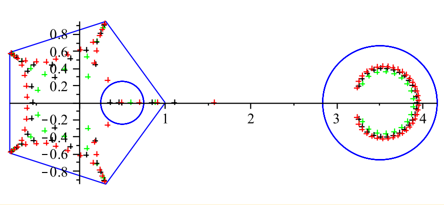
$$\mu = \mu^{G_1} + \mu^{G_2} - \mu^{K}$$



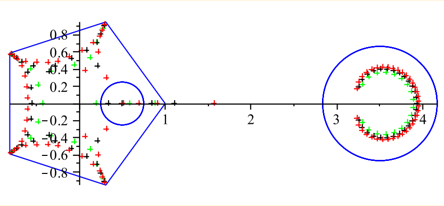
$$\mu = \mu^{G_1} + \mu^{G_2} + \mu^{K}$$

A very suggestive example...

Zeros of p_{40}, p_{60} and p_{80}, for pentagon G_1, disk lake K, disk G_2



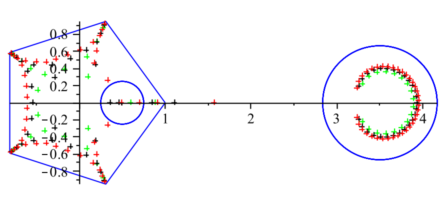
$$\mu = \mu^{G_1} + \mu^{G_2} - \mu^{K}$$



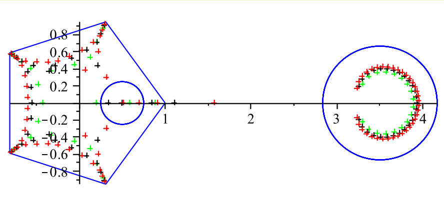
$$\mu = \mu^{G_1} + \mu^{G_2} + \mu^{K}$$

A very suggestive example...

Zeros of p_{40}, p_{60} and p_{80}, for pentagon G_1, disk lake K, disk G_2



$$\mu = \mu^{G_1} + \mu^{G_2} - \mu^{K}$$



$$\mu = \mu^{G_1} + \mu^{G_2} + \mu^{K}$$