

NUMERICAL ANALYSIS OF THE MFS FOR HARMONIC PROBLEMS IN ANNULAR DOMAINS

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ABSTRACT. In this study, we investigate the application of the Method of Fundamental Solutions (MFS) to the Dirichlet problem for Laplace's equation in an annular domain. We examine the properties of the resulting coefficient matrix and its eigenvalues. The convergence of the method is proved for analytic boundary data. An efficient matrix decomposition algorithm using Fast Fourier Transforms (FFT's) is developed for the computation of the MFS approximation. We also tested the algorithm numerically on several problems confirming the theoretical predictions.

1. METHOD OF FUNDAMENTAL SOLUTIONS FORMULATION

We consider the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f_1 & \text{on } \partial\Omega_1, \\ u = f_2 & \text{on } \partial\Omega_2, \end{cases} \quad (1.1)$$

where the domain Ω is the annulus of radii ϱ_1 and ϱ_2 ,

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 : \varrho_1 < |\mathbf{x}| < \varrho_2\}, \quad (1.2)$$

Δ denotes the Laplace operator and f_1 and f_2 are given functions. The boundary of Ω is $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ where $\partial\Omega_1$ and $\partial\Omega_2$ are the circles with radii ϱ_1 and ϱ_2 , respectively. Let the function $k(P, Q)$ be a fundamental solution of Laplace's equation given by

$$k(P, Q) = -\frac{1}{2\pi} \log |P - Q|, \quad (1.3)$$

with $|P - Q|$ denoting the distance between the points P and Q . In the Method of Fundamental Solutions (MFS), the solution u is approximated by

$$u_N(\mathbf{c}, \mathbf{Q}; P) = \sum_{j=1}^{2N} c_j k(P, Q_j), \quad P \in \bar{\Omega}, \quad (1.4)$$

where $\mathbf{c} = (c_1, c_2, \dots, c_{2N})^T$ and \mathbf{Q} is a $4N$ -vector containing the coordinates of the singularities Q_j , $j = 1, \dots, 2N$, which lie outside $\bar{\Omega}$. The singularities Q_j are fixed on the boundary $\partial\tilde{\Omega} = \partial\tilde{\Omega}_1 \cup \partial\tilde{\Omega}_2$ of an annulus $\tilde{\Omega}$ concentric to Ω and defined by

$$\tilde{\Omega} = \{\mathbf{x} \in \mathbb{R}^2 : R_1 < |\mathbf{x}| < R_2\},$$

where $R_2 > \varrho_2 > \varrho_1 > R_1$. The boundary of $\tilde{\Omega}$ comprises $\partial\tilde{\Omega}_1$ and $\partial\tilde{\Omega}_2$, the circles with radii R_1 and R_2 , respectively. A set of collocation points $\{P_i\}_{i=1}^{2N}$ is placed on $\partial\tilde{\Omega}$. If $P_i = (x_{P_i}, y_{P_i})$, then we take

$$x_{P_i} = \varrho_1 \cos \frac{2(i-1)\pi}{N}, \quad y_{P_i} = \varrho_1 \sin \frac{2(i-1)\pi}{N}, \quad i = 1, \dots, N, \quad (1.5)$$

and

$$x_{P_{N+i}} = \varrho_2 \cos \frac{2(i-1)\pi}{N}, \quad y_{P_{N+i}} = \varrho_2 \sin \frac{2(i-1)\pi}{N}, \quad i = 1, \dots, N. \quad (1.6)$$

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If $Q_j = (x_{Q_j}, y_{Q_j})$, then

$$x_{Q_j} = R_1 \cos \frac{2(j-1+\alpha)\pi}{N}, \quad y_{Q_j} = R_1 \sin \frac{2(j-1+\alpha)\pi}{N}, \quad j = 1, \dots, N, \quad (1.7)$$

and

$$x_{Q_{N+j}} = R_2 \cos \frac{2(j-1+\alpha)\pi}{N}, \quad y_{Q_{N+j}} = R_2 \sin \frac{2(j-1+\alpha)\pi}{N}, \quad j = 1, \dots, N. \quad (1.8)$$

The presence of the angular parameter $\alpha \in [0, 1)$ indicates that the sources are rotated by an angle $\frac{2\pi\alpha}{N}$ from the boundary points (see [9]). In the MFS, the coefficients \mathbf{c} are determined so that the boundary condition is satisfied at the boundary points $\{P_i\}_{i=1}^{2N}$:

$$u_N(\mathbf{c}, \mathbf{Q}; P_i) = f_1(P_i), \quad u_N(\mathbf{c}, \mathbf{Q}; P_{N+i}) = f_2(P_{N+i}), \quad i = 1, \dots, N.$$

This yields a linear system of the form

$$\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}, \quad (1.9)$$

where

$$\begin{aligned} \mathbf{f}_1 &= (f_1(P_1), f_1(P_2), \dots, f_1(P_N))^T, & \mathbf{f}_2 &= (f_2(P_{N+1}), f_2(P_{N+2}), \dots, f_2(P_{2N}))^T, \\ \mathbf{d}_1 &= (c_1, c_2, \dots, c_N)^T, & \mathbf{d}_2 &= (c_{N+1}, c_{N+2}, \dots, c_{2N})^T. \end{aligned}$$

The elements of the $N \times N$ submatrices A_{11} , A_{12} , A_{21} and A_{22} are given by

$$\begin{aligned} (A_{11})_{i,j} &= -\frac{1}{2\pi} \log |P_i - Q_j|, & (A_{12})_{i,j} &= -\frac{1}{2\pi} \log |P_i - Q_{N+j}|, \\ (A_{21})_{i,j} &= -\frac{1}{2\pi} \log |P_{N+i} - Q_j|, & (A_{22})_{i,j} &= -\frac{1}{2\pi} \log |P_{N+i} - Q_{N+j}|, \end{aligned}$$

respectively, for $i, j = 1, \dots, N$. The matrices A_{11} , A_{12} , A_{21} and A_{22} are circulant. An extensive account of the properties of circulant matrices can be found in [1]. Details on the MFS and its applications can be found in the survey articles [2, 3, 4]. The convergence of the MFS for harmonic problems in the disk subject to Dirichlet boundary conditions has been studied in [6, 7, 10]. The convergence of a modified MFS for annular domains was studied by Katsurada [6] using Green's functions instead of the usual fundamental solutions, and considering two different circular problems. Our approach is different as it uses the standard fundamental solutions directly in the annulus and is based on the study of the eigenvalues of the coefficient matrix.

The paper is structured as follows: In Section 2, we formulate the matrix decomposition algorithm for the solution of the linear system (1.9). In Section 3, we develop the properties of the eigenvalues of the coefficient matrix and study its invertibility. An explicit expression for the MFS approximation is then given in Section 4. The convergence of the method for analytic boundary data is proved in Section 5. Note that the proof of the convergence is rather technical and for the sake brevity, many details are omitted but can be found in [?]. Numerical experiments which enable us to confirm the theoretical predictions are given in Section 6. Finally, some conclusions are presented in Section 7.

2. MATRIX DECOMPOSITION ALGORITHM

In order to develop a matrix decomposition algorithm, we need the following fact [1]. If a matrix A is circulant, written as $A = \text{circ}(a_1, \dots, a_N)$, then it can be diagonalized as $A = U^*DU$, where U is the

unitary $N \times N$ Fourier matrix with conjugate

$$U^* = \frac{1}{N^{1/2}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix},$$

where $\omega = e^{2\pi i/N}$, $D = \text{diag}(\lambda_1, \dots, \lambda_N)$, and

$$\lambda_j = \sum_{k=1}^N a_k \omega^{(k-1)(j-1)}, \quad j = 1, \dots, N, \quad (2.1)$$

are the eigenvalues of A . The eigenvector corresponding to λ_j is

$$\boldsymbol{\xi}_j = \frac{1}{N^{1/2}} \left(1, \omega^{(j-1)}, \dots, \omega^{(N-1)(j-1)} \right)^T.$$

Let $\langle \boldsymbol{\zeta}, \boldsymbol{\eta} \rangle = \sum_{k=1}^N \zeta_k \bar{\eta}_k$ be the inner product of $\boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{C}^N$. Since $\{\boldsymbol{\xi}_j\}_{j=1, \dots, N}$ forms an orthonormal basis of \mathbb{C}^N , any vector $\mathbf{v} \in \mathbb{C}^N$ can be expressed as $\mathbf{v} = \sum_{k=1}^N \langle \mathbf{v}, \boldsymbol{\xi}_k \rangle \boldsymbol{\xi}_k$ and hence $A\mathbf{v} = \sum_{k=1}^N d_k \langle \mathbf{v}, \boldsymbol{\xi}_k \rangle \boldsymbol{\xi}_k$. When A is nonsingular

$$A^{-1}\mathbf{v} = \sum_{k=1}^N \frac{1}{d_k} \langle \mathbf{v}, \boldsymbol{\xi}_k \rangle \boldsymbol{\xi}_k. \quad (2.2)$$

The system (1.9) can therefore be written as

$$\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array} \right) \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) \left(\begin{array}{c|c} U^* & 0 \\ \hline 0 & U^* \end{array} \right) \left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array} \right) \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} = \left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array} \right) \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix} \quad (2.3)$$

or

$$\left(\begin{array}{c|c} UA_{11}U^* & UA_{12}U^* \\ \hline UA_{21}U^* & UA_{22}U^* \end{array} \right) \begin{pmatrix} U\mathbf{d}_1 \\ U\mathbf{d}_2 \end{pmatrix} = \begin{pmatrix} U\mathbf{f}_1 \\ U\mathbf{f}_2 \end{pmatrix} \quad (2.4)$$

or

$$\left(\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array} \right) \begin{pmatrix} \hat{\mathbf{d}}_1 \\ \hat{\mathbf{d}}_2 \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{f}}_1 \\ \hat{\mathbf{f}}_2 \end{pmatrix}, \quad (2.5)$$

where

$$\hat{\mathbf{d}}_1 = U\mathbf{d}_1, \quad \hat{\mathbf{d}}_2 = U\mathbf{d}_2, \quad \hat{\mathbf{f}}_1 = U\mathbf{f}_1, \quad \hat{\mathbf{f}}_2 = U\mathbf{f}_2,$$

and

$$D_{ij} = \text{diag}(\lambda_1^{ij}, \dots, \lambda_N^{ij}), \quad i, j = 1, 2$$

is the diagonal matrix whose diagonal elements are the eigenvalues of the submatrix A_{ij} , $i, j = 1, 2$. The solution of system (2.5) can thus be reduced to the solution of the N independent 2×2 systems

$$\left(\begin{array}{c|c} \lambda_k^{11} & \lambda_k^{12} \\ \hline \lambda_k^{21} & \lambda_k^{22} \end{array} \right) \begin{pmatrix} \hat{d}_k^1 \\ \hat{d}_k^2 \end{pmatrix} = \begin{pmatrix} \hat{f}_k^1 \\ \hat{f}_k^2 \end{pmatrix}, \quad k = 1, 2, \dots, N, \quad (2.6)$$

from which it follows that

$$\hat{d}_k^1 = \frac{\lambda_k^{22} \hat{f}_k^1 - \lambda_k^{12} \hat{f}_k^2}{\lambda_k^{11} \lambda_k^{22} - \lambda_k^{12} \lambda_k^{21}}, \quad \hat{d}_k^2 = \frac{-\lambda_k^{21} \hat{f}_k^1 + \lambda_k^{11} \hat{f}_k^2}{\lambda_k^{11} \lambda_k^{22} - \lambda_k^{12} \lambda_k^{21}} \quad k = 1, \dots, N.$$

Having obtained $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$, we can determine \mathbf{d}_1 and \mathbf{d}_2 (and hence \mathbf{c}) from

$$\mathbf{d}_1 = U^* \hat{\mathbf{d}}_1, \quad \mathbf{d}_2 = U^* \hat{\mathbf{d}}_2. \quad (2.7)$$

We thus have the following matrix decomposition algorithm for solving (1.9):

Algorithm

Step 1. Compute $\hat{f}_1 = Uf_1$ and $\hat{f}_2 = Uf_2$.

Step 2. Construct the diagonal matrices D_{ij} , $i, j = 1, 2$.

Step 3. Evaluate \hat{d}_1 and \hat{d}_2 .

Step 4. Compute $d_1 = U^*\hat{d}_1$, $d_2 = U^*\hat{d}_2$.

Remarks

- (i) In **Step 1** and **Step 4**, because of the form of the matrices U and U^* , the operations can be carried out via Fast Fourier Transforms (FFTs) at a cost of $O(N \log N)$ operations.
- (ii) FFTs can also be used for the evaluation of the diagonal matrices in **Step 2**.
- (iii) The FFT operations are performed using the NAG [8] routines C06FPF, C06FQF and C06FRF.
- (iv) The same algorithm is also applicable when, instead of the boundary conditions of (1.1), we have

$$u = f_1 \quad \text{on} \quad \partial\Omega_1, \quad \frac{\partial u}{\partial n} = g_2 \quad \text{on} \quad \partial\Omega_2,$$

or

$$\frac{\partial u}{\partial n} = g_1 \quad \text{on} \quad \partial\Omega_1, \quad u = f_2 \quad \text{on} \quad \partial\Omega_2.$$

3. PROPERTIES OF THE EIGENVALUES

Since our aim is to solve (1.9), we investigate the properties of the eigenvalues of the matrix

$$G = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right).$$

In particular, we are interested in the cases where these eigenvalues vanish. We first consider the eigenvalues of the submatrices A_{11} , A_{12} , A_{21} , A_{22} . In the following, we denote by A each of the matrices A_{ij} , $i, j = 1, 2$, and by λ_k each of the eigenvalues λ_k^{ij} , $i, j = 1, 2$. Thus we shall denote by (a_1, \dots, a_N) the vector $(a_1^{ij}, \dots, a_N^{ij})$, which generates the circulant matrix A_{ij} , $i, j = 1, 2$. Since each of the A_{ij} 's involves the distances between points on two concentric circles, we shall denote the radii of these circles by R and ϱ . The circle of radius R is part of $\partial\Omega$ while the circle of radius ϱ is part of $\partial\Omega$.

We divide these eigenvalues into three groups:

- (i) the 'first' eigenvalue λ_1 ;
- (ii) when N is even, the eigenvalue $\lambda_{\frac{N}{2}+1}$;
- (iii) the remaining eigenvalues.

3.1. The first eigenvalue. We can derive an exact expression for the first eigenvalue:

Proposition 3.1. *The eigenvalue λ_1 is given by*

$$\lambda_1 = -\frac{1}{4\pi} \log (R^{2N} - 2R^N \varrho^N \cos(2\pi\alpha) + \varrho^{2N}). \quad (3.1)$$

Proof. Since A is circulant, from (2.1), we have

$$\lambda_1 = \sum_{k=1}^N a_k. \quad (3.2)$$

Since

$$a_k = -\frac{1}{4\pi} \log \left(R^2 + \varrho^2 - 2R\varrho \cos \left(\frac{2\pi(k + \alpha - 1)}{N} \right) \right), \quad (3.3)$$

it follows that

$$\lambda_1 = -\frac{1}{4\pi} \log \left\{ \prod_{k=1}^N \left(R^2 + \varrho^2 - 2R\varrho \cos \left(\frac{2\pi(k + \alpha - 1)}{N} \right) \right) \right\}.$$

From the identity [5, p. 40]

$$\prod_{k=0}^{n-1} \left\{ x^2 - 2xy \cos \left(\alpha + \frac{2k\pi}{n} \right) + y^2 \right\} = x^{2n} - 2x^n y^n \cos n\alpha + y^{2n}, \quad (3.4)$$

(3.1) follows. \square

We have the following corollary:

Corollary 3.2. *For $\alpha = 0$, the matrix A is nonsingular if and only if $R^N - \varrho^N = 1$.* \square

3.2. The eigenvalue $\lambda_{N/2+1}$ when N is even. We next investigate the behaviour of the eigenvalue $\lambda_{N/2+1}$ in the case N is even. In particular, we have the exact expression:

Proposition 3.3. *If N is even, then*

$$\lambda_{N/2+1} = -\frac{1}{4\pi} \log \frac{R^N - 2R^{N/2}\varrho^{N/2} \cos \alpha\pi + \varrho^N}{R^N - 2R^{N/2}\varrho^{N/2} \cos(\alpha+1)\pi + \varrho^N}. \quad (3.5)$$

Proof. From (2.1), the eigenvalues of A are given by

$$\lambda_j = \sum_{k=1}^N a_k \omega^{(k-1)(j-1)}, \quad j = 1, 2, \dots, N,$$

and therefore

$$\begin{aligned} \lambda_{N/2+1} &= -\frac{1}{4\pi} \sum_{k=1}^N (-1)^{k-1} \log \left(R^2 - 2R\varrho \cos \frac{2\pi(k + \alpha - 1)}{N} + \varrho^2 \right) \\ &= -\frac{1}{4\pi} \sum_{n=1}^{N/2} \log \left(R^2 - 2R\varrho \cos \left(2\pi \frac{n-1}{M} + \frac{\alpha\pi}{M} \right) + \varrho^2 \right) \\ &\quad + \frac{1}{4\pi} \sum_{n=1}^{N/2} \log \left(R^2 - 2R\varrho \cos \left(2\pi \frac{n-1}{M} + \frac{(\alpha+1)\pi}{M} \right) + \varrho^2 \right). \end{aligned}$$

From (3.4) we obtain

$$\begin{aligned} \lambda_{N/2+1} &= -\frac{1}{4\pi} \log \left(R^{2M} - 2R^M \varrho^M \cos \alpha\pi + \varrho^{2M} \right) \\ &\quad + \frac{1}{4\pi} \log \left(R^{2M} - 2R^M \varrho^M \cos(\alpha+1)\pi + \varrho^{2M} \right) \\ &= -\frac{1}{4\pi} \log \frac{R^N - 2R^{N/2}\varrho^{N/2} \cos \alpha\pi + \varrho^N}{R^N - 2R^{N/2}\varrho^{N/2} \cos(\alpha+1)\pi + \varrho^N}. \end{aligned}$$

\square

A direct application of this proposition yields the following corollary:

Corollary 3.4. *For N even, the eigenvalue $\lambda_{N/2+1} = 0$ if and only if $\alpha = \frac{1}{2}$.*

Proof. Taking $\alpha = \frac{1}{2}$ in (3.5) yields that $\lambda_{\frac{N}{2}+1} = 0$. For $\alpha \in [0, \frac{1}{2})$, it is clear that $\cos \alpha\pi > 0$ whereas $\cos(\alpha+1)\pi < 0$, and (3.5) yields that in this case $\lambda_{\frac{N}{2}+1} \neq 0$. \square

3.3. The eigenvalues λ_j , $j \neq 1, N/2 + 1$, when N is even. In this case we have the expression:

Proposition 3.5. *For $\alpha \in [0, \frac{1}{2}]$ and $j \neq 1$, we have*

$$\lambda_j = \frac{N}{4\pi} \sum_{m=0}^{\infty} \left\{ \frac{e^{-i\frac{2\pi}{N}(mN+j-1)\alpha} \varrho^{mN+j-1}}{(mN+j-1)R^{mN+j-1}} + \frac{e^{i\frac{2\pi}{N}((m+1)N-j+1)\alpha} \varrho^{(m+1)N-j+1}}{((m+1)N-j+1)R^{(m+1)N-j+1}} \right\}. \quad (3.6)$$

In particular when N is even

$$\lambda_{\frac{N}{2}+1} = \frac{N}{2\pi} \sum_{m=0}^{\infty} \frac{\cos((2m+1)\alpha\pi) \varrho^{mN+\frac{N}{2}}}{(mN+\frac{N}{2}) R^{mN+\frac{N}{2}}}. \quad (3.7)$$

Proof. From [5, p. 52], we have that

$$F(R, \varrho, \vartheta) = -\frac{1}{4\pi} \log(R^2 - 2R\varrho \cos \vartheta + \varrho^2) = -\frac{1}{2\pi} \log R + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\varrho^n \cos n\vartheta}{nR^n},$$

where $R > \varrho$. This is true in the cases where $R = R_2, \varrho = \varrho_1$ and $R = R_2, \varrho = \varrho_2$. When $R < \varrho$, i.e., when $R = R_1, \varrho = \varrho_1$ and $R = R_1, \varrho = \varrho_2$, in a similar way, we have

$$F(\varrho, R, \vartheta) = -\frac{1}{4\pi} \log(R^2 - 2R\varrho \cos \vartheta + \varrho^2) = -\frac{1}{2\pi} \log \varrho + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{R^n \cos n\vartheta}{n\varrho^n}.$$

Then, from (2.1), for $j = 2, \dots, N$,

$$\begin{aligned} \lambda_j &= \sum_{k=1}^N e^{i\frac{2\pi}{N}(j-1)(k-1)} F\left(R, \varrho, \frac{2\pi}{N}(k-1+\alpha)\right) \\ &= \frac{1}{2\pi} \sum_{k=1}^N \cos\left(\frac{2\pi}{N}(j-1)(k-1)\right) \left\{ \sum_{n=1}^{\infty} \frac{\varrho^n \cos\left(\frac{2\pi}{N}n(k-1+\alpha)\right)}{nR^n} \right\} \\ &\quad + i \frac{1}{2\pi} \sum_{k=1}^N \sin\left(\frac{2\pi}{N}(j-1)(k-1)\right) \left\{ \sum_{n=1}^{\infty} \frac{\varrho^n \cos\left(\frac{2\pi}{N}n(k-1+\alpha)\right)}{nR^n} \right\} \\ &= \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\varrho^n}{nR^n} \left\{ \sum_{k=1}^N \cos\left(\frac{2\pi}{N}(j-1)(k-1)\right) \cos\left(\frac{2\pi}{N}n(k-1+\alpha)\right) \right\} \\ &\quad + i \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\varrho^n}{nR^n} \left\{ \sum_{k=1}^N \sin\left(\frac{2\pi}{N}(j-1)(k-1)\right) \cos\left(\frac{2\pi}{N}n(k-1+\alpha)\right) \right\}. \end{aligned} \quad (3.8)$$

For the first sum, we have from [10],

$$\sum_{k=1}^N \cos\left(\frac{2\pi}{N}k(j-1)\right) \cos\left(\frac{2\pi}{N}n(k-1+\alpha)\right) = \frac{1}{2} \cos\left(\frac{2\pi}{N}n\alpha\right) \{C_{n-j+1}^N + C_{n+j-1}^N\}, \quad (3.9)$$

$$\sum_{k=1}^N \sin\left(\frac{2\pi}{N}k(j-1)\right) \cos\left(\frac{2\pi}{N}n(k-1+\alpha)\right) = \frac{1}{2} \sin\left(\frac{2\pi}{N}n\alpha\right) \{C_{n-j+1}^N - C_{n+j-1}^N\}, \quad (3.10)$$

where

$$C_{\kappa}^N = \sum_{j=1}^N \cos\left(\frac{2\pi}{N}\kappa j\right) = \begin{cases} N & \text{if } \kappa = 0 \pmod{N}, \\ 0 & \text{if } \kappa \neq 0 \pmod{N}. \end{cases} \quad (3.11)$$

Combining (3.8), (3.9) and (3.10), we obtain

$$\begin{aligned} \lambda_j &= \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{\varrho^n \cos\left(\frac{2\pi}{N}n\alpha\right)}{nR^n} \{C_{n-j+1}^N + C_{n+j-1}^N\} - \frac{i}{4\pi} \sum_{n=1}^{\infty} \frac{\varrho^n \sin\left(\frac{2\pi}{N}n\alpha\right)}{nR^n} \{C_{n-j+1}^N - C_{n+j-1}^N\} \\ &= \frac{N}{4\pi} \sum_{m=0}^{\infty} \left\{ \frac{\varrho^{mN+j-1} e^{-i\frac{2\pi}{N}(mN+j-1)\alpha}}{(mN+j-1)R^{mN+j-1}} + \frac{e^{i\frac{2\pi}{N}(mN+N-j+1)\alpha} \varrho^{mN+N-j+1}}{(mN+N-j+1)R^{mN+N-j+1}} \right\} \end{aligned}$$

which proves (3.6). Formula (3.7) is an immediate consequence of (3.6). \square

We also have the following two corollaries when $\alpha = 0$:

Corollary 3.6. *For $\alpha = 0$ and $j \neq 1$, we have*

$$\lambda_j = \frac{N}{4\pi} \sum_{m=0}^{\infty} \left(\frac{1}{j-1+mN} \cdot \frac{\varrho^{j-1+mN}}{R^{j-1+mN}} + \frac{1}{N-j+1+mN} \cdot \frac{\varrho^{N-j+1+mN}}{R^{N-j+1+mN}} \right). \quad (3.12)$$

In particular, for N even and $j = \frac{N}{2} + 1$,

$$\lambda_{\frac{N}{2}+1} = \frac{N}{2\pi} \sum_{m=0}^{\infty} \frac{1}{\frac{N}{2} + mN} \cdot \frac{\varrho^{\frac{N}{2}+mN}}{R^{\frac{N}{2}+mN}}. \quad (3.13)$$

Also, for $j = \lfloor \frac{N}{2} \rfloor + 1, \dots, N$, we have $\lambda_j = \lambda_{N-j+2}$.

Corollary 3.7. *For $j = 2, \dots, N$, we have $\lambda_j > 0$.* \square

3.4. Invertibility of the matrix G . Since

$$G = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) = \left(\begin{array}{c|c} U^* & 0 \\ \hline 0 & U^* \end{array} \right) \left(\begin{array}{c|c} D_{11} & D_{12} \\ \hline U & 0 \\ 0 & U \end{array} \right),$$

it is sufficient to examine the matrix $D = \left(\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array} \right)$. The following lemma can be easily proved.

Lemma 3.8. *If $\det D$ denotes the determinant of D , then*

$$\det D = \det \left(\begin{array}{cccc|cccc} \lambda_1^{11} & 0 & 0 & \cdots & 0 & \lambda_1^{12} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^{11} & 0 & \cdots & 0 & 0 & \lambda_2^{12} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^{11} & \cdots & 0 & 0 & 0 & \lambda_3^{12} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_N^{11} & 0 & 0 & 0 & \cdots & \lambda_N^{12} \\ \lambda_1^{21} & 0 & 0 & \cdots & 0 & \lambda_1^{22} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^{21} & 0 & \cdots & 0 & 0 & \lambda_2^{22} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^{21} & \cdots & 0 & 0 & 0 & \lambda_3^{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_N^{21} & 0 & 0 & 0 & \cdots & \lambda_N^{22} \end{array} \right) = \prod_{k=1}^N (\lambda_k^{11} \lambda_k^{22} - \lambda_k^{12} \lambda_k^{21}). \quad (3.14)$$

Therefore, the matrix D is nonsingular if and only if

$$\lambda_k^{11} \lambda_k^{22} - \lambda_k^{12} \lambda_k^{21} \neq 0, \quad k = 1, \dots, N. \quad \square$$

When $\alpha = 0$ we have the following theorem.

Theorem 3.9. *For $\alpha = 0$, the matrix G is singular if and only if*

$$\lambda_1^{11}\lambda_1^{22} - \lambda_1^{12}\lambda_1^{21} = 0.$$

Proof. Let

$$F_{N,j}(\omega) = \frac{N}{4\pi} \sum_{m=0}^{\infty} \left(\frac{\omega^{j-1+mN}}{j-1+mN} + \frac{\omega^{N-j+1+mN}}{N-j+1+mN} \right).$$

Clearly $F_{N,j}(\omega)$, $\omega \in (0, 1)$, is positive and strictly increasing. For $j \neq 1$, we have (see Corollary 3.6)

$$\lambda_j^{11} = F_{N,j}\left(\frac{R_1}{\varrho_1}\right), \lambda_j^{12} = F_{N,j}\left(\frac{\varrho_1}{R_2}\right), \lambda_j^{21} = F_{N,j}\left(\frac{R_1}{\varrho_2}\right) \text{ and } \lambda_j^{22} = F_{N,j}\left(\frac{\varrho_2}{R_2}\right).$$

Since $R_1 < \varrho_1 < \varrho_2 < R_2$, we have

$$0 < \frac{R_1}{\varrho_2} < \frac{R_1}{\varrho_1}, \text{ so that } \lambda_j^{21} < \lambda_j^{11},$$

and

$$0 < \frac{\varrho_1}{R_2} < \frac{\varrho_2}{R_2}, \text{ so that } \lambda_j^{12} < \lambda_j^{22}.$$

Thus,

$$\lambda_k^{11} \cdot \lambda_k^{22} - \lambda_k^{12} \cdot \lambda_k^{21} > 0, \quad k = 2, 3, \dots, N.$$

Therefore, the only case in which the matrix G can be singular is when $\lambda_1^{11}\lambda_1^{22} - \lambda_1^{12}\lambda_1^{21} = 0$. \square

4. EXPLICIT EXPRESSION FOR THE APPROXIMATE SOLUTION IN TERMS OF EIGENVALUES AND EIGENVECTORS.

Denoting the approximation $u_N(\mathbf{c}, \mathbf{Q}; P)$ in (1.4) by $u_N(P)$, we have

$$u_N(P) = \sum_{k=1}^N d_k^1 \log |P - Q_k| + \sum_{k=1}^N d_k^2 \log |P - Q_{N+k}| = \langle \mathbf{d}_1, \mathbf{l}_1 \rangle + \langle \mathbf{d}_2, \mathbf{l}_2 \rangle, \quad (4.1)$$

where

$$\mathbf{l}_1 = (\log |P - Q_1|, \dots, \log |P - Q_N|)^T, \quad \mathbf{l}_2 = (\log |P - Q_{N+1}|, \dots, \log |P - Q_{2N}|)^T$$

and $\mathbf{d}_1, \mathbf{d}_2$ satisfy (1.9). Thus

$$\begin{aligned} \mathbf{d}_1 &= \left(A_{12}^{-1} A_{11} - A_{22}^{-1} A_{21} \right)^{-1} \left(A_{12}^{-1} \mathbf{f}_1 - A_{22}^{-1} \mathbf{f}_2 \right), \\ \mathbf{d}_2 &= \left(A_{11}^{-1} A_{12} - A_{21}^{-1} A_{22} \right)^{-1} \left(A_{11}^{-1} \mathbf{f}_1 - A_{21}^{-1} \mathbf{f}_2 \right), \end{aligned}$$

so that

$$\begin{aligned} \langle \mathbf{d}_2, \mathbf{l}_2 \rangle &= \left\langle \left(A_{11}^{-1} A_{12} - A_{21}^{-1} A_{22} \right)^{-1} \left(A_{11}^{-1} \sum_{k=1}^N \langle \mathbf{f}_1, \boldsymbol{\xi}_k \rangle \boldsymbol{\xi}_k - A_{21}^{-1} \sum_{k=1}^N \langle \mathbf{f}_2, \boldsymbol{\xi}_k \rangle \boldsymbol{\xi}_k \right), \mathbf{l}_2 \right\rangle \\ &= \sum_{k=1}^N \frac{\lambda_k^{11}}{\Delta_k} \langle \mathbf{f}_2, \boldsymbol{\xi}_k \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_k \rangle} - \sum_{k=1}^N \frac{\lambda_k^{21}}{\Delta_k} \langle \mathbf{f}_1, \boldsymbol{\xi}_k \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_k \rangle} \end{aligned}$$

where $\Delta_k = \lambda_k^{11}\lambda_k^{22} - \lambda_k^{12}\lambda_k^{21}$. We assume that the matrix inverses appearing in the above equations exist. Analogously,

$$\langle \mathbf{d}_1, \mathbf{l}_1 \rangle = \sum_{k=1}^N \frac{\lambda_k^{22}}{\Delta_k} \langle \mathbf{f}_1, \boldsymbol{\xi}_k \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_k \rangle} - \sum_{k=1}^N \frac{\lambda_k^{12}}{\Delta_k} \langle \mathbf{f}_2, \boldsymbol{\xi}_k \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_k \rangle}.$$

Therefore, we obtain the explicit expression for the MFS approximation,

$$\begin{aligned}
 u_N(P) &= \sum_{k=1}^N \frac{\lambda_k^{11}}{\Delta_k} \langle \mathbf{f}_2, \boldsymbol{\xi}_k \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_k \rangle} - \sum_{k=1}^N \frac{\lambda_k^{21}}{\Delta_k} \langle \mathbf{f}_1, \boldsymbol{\xi}_k \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_k \rangle} \\
 &\quad - \sum_{k=1}^N \frac{\lambda_k^{12}}{\Delta_k} \langle \mathbf{f}_2, \boldsymbol{\xi}_k \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_k \rangle} + \sum_{k=1}^N \frac{\lambda_k^{22}}{\Delta_k} \langle \mathbf{f}_1, \boldsymbol{\xi}_k \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_k \rangle}.
 \end{aligned} \tag{4.2}$$

Note that the evaluation of the approximate solution via equation (4.2) is equivalent to applying the matrix decomposition algorithm described in Section 2.

5. CONVERGENCE OF THE MFS FOR ANALYTIC BOUNDARY DATA

In this section, we show that the MFS approximation u_N converges exponentially with respect to N to the exact solution u of (1.1) in the $\|\cdot\|_\infty$ -norm, provided the boundary data f_1, f_2 are analytic, or, equivalently, u can be extended as a harmonic function in an open region V containing the annulus.

The analyticity of the boundary data yields that there exists a constant $\beta > 0$, such that

$$\overline{\Omega}_\beta = \{ \mathbf{x} \in \mathbb{R}^2 \mid \varrho_1 - \beta \leq |\mathbf{x}| \leq \varrho_2 + \beta \} \subset V. \tag{5.1}$$

The analytic solution u , expressed in polar coordinates, is of the form [12]

$$u(r, \vartheta) = \sum_{k \in \mathbb{Z}} c_k r^k e^{ik\vartheta} + \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} d_k r^k e^{-ik\vartheta} + d_0 \log r. \tag{5.2}$$

Thus (5.2) implies that the boundary data can be expressed as

$$f_j(\vartheta) = \sum_{k \in \mathbb{Z}} c_k \varrho_j^k e^{ik\vartheta} + \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} d_k \varrho_j^k e^{-ik\vartheta} + d_0 \log \varrho_j, \quad j = 1, 2.$$

The analyticity assumption also implies that there exists a positive constant M_β such that

$$|c_k|, |d_k| \leq M_\beta (\varrho_2 + \beta)^{-k} \text{ for } k \geq 0, \tag{5.3}$$

$$|c_k|, |d_k| \leq M_\beta (\varrho_1 - \beta)^k \text{ for } k < 0. \tag{5.4}$$

For simplicity, we prove the convergence of the MFS approximation to the exact solution in the case when $\alpha = 0$.

5.1. The discrete Fourier interpolant. We assume that N is even. (The case when N is odd can be treated analogously.) We define the *discrete Fourier interpolant* \tilde{u}_N , to be

$$\tilde{u}_N(r, \vartheta) = \sum_{k=-N/2+1}^{N/2} \mu_k r^k e^{ik\vartheta} + \sum_{\substack{k=-N/2+1 \\ k \neq 0}}^{N/2} \nu_k r^{-k} e^{ik\vartheta} + \nu_0 \log r. \tag{5.5}$$

We also define

$$f_1^N(\vartheta) = \tilde{u}_N(\varrho_1, \vartheta), \quad f_2^N(\vartheta) = \tilde{u}_N(\varrho_2, \vartheta).$$

The coefficients $\mu_k, \nu_k \in \mathbb{C}$, $k = -\frac{N}{2} + 1, \dots, \frac{N}{2}$, are chosen so that \tilde{u}_N agrees with the boundary data f_1, f_2 at the points $\varrho_1 e^{i\vartheta_j}, \varrho_2 e^{i\vartheta_j}$, where $\vartheta_j = 2\pi(j-1)/N$, $j = 1, \dots, N$. Thus for $k = -\frac{N}{2} + 1, \dots, \frac{N}{2}$

with $k \neq 0$, the μ_k, ν_k satisfy

$$\mu_k \varrho_1^k + \nu_k \varrho_1^{-k} = \sum_{m \in \mathbb{Z}} (c_{k+mN} \varrho_1^{k+mN} + d_{-k-mN} \varrho_1^{-k-mN}),$$

$$\mu_k \varrho_2^k + \nu_k \varrho_2^{-k} = \sum_{m \in \mathbb{Z}} (c_{k+mN} \varrho_2^{k+mN} + d_{-k-mN} \varrho_2^{-k-mN}),$$

from which it follows that, with $\delta_k = \varrho_1^k \varrho_2^{-k} - \varrho_1^{-k} \varrho_2^k$,

$$\mu_k = \frac{\varrho_2^{-k} \sum_{m \in \mathbb{Z}} (c_{k+mN} \varrho_1^{k+mN} + d_{-k-mN} \varrho_1^{-k-mN})}{\varrho_1^k \varrho_2^{-k} - \varrho_1^{-k} \varrho_2^k} - \frac{\varrho_1^{-k} \sum_{m \in \mathbb{Z}} (c_{k+mN} \varrho_2^{k+mN} + d_{-k-mN} \varrho_2^{-k-mN})}{\delta_k}, \quad (5.6)$$

$$\nu_k = \frac{\varrho_1^k \sum_{m \in \mathbb{Z}} (c_{k+mN} \varrho_2^{k+mN} + d_{-k-mN} \varrho_2^{-k-mN})}{\delta_k} - \frac{\varrho_2^k \sum_{m \in \mathbb{Z}} (c_{k+mN} \varrho_1^{k+mN} + d_{-k-mN} \varrho_1^{-k-mN})}{\delta_k}. \quad (5.7)$$

For $k = 0$, we have

$$\mu_0 + \nu_0 \log \varrho_1 = \sum_{m \in \mathbb{Z}} c_{mN} \varrho_1^{mN} + \sum_{m \in \mathbb{Z}, m \neq 0} d_{mN} \varrho_1^{mN} + d_0 \log \varrho_1,$$

$$\mu_0 + \nu_0 \log \varrho_2 = \sum_{m \in \mathbb{Z}} c_{mN} \varrho_2^{mN} + \sum_{m \in \mathbb{Z}, m \neq 0} d_{mN} \varrho_2^{mN} + d_0 \log \varrho_2,$$

and therefore

$$\mu_0 = \frac{1}{\log \varrho_2 / \varrho_1} \left\{ \log \varrho_2 \left[\sum_{m \in \mathbb{Z}} c_{mN} \varrho_1^{mN} + \sum_{m \in \mathbb{Z}, m \neq 0} d_{mN} \varrho_1^{mN} \right] - \log \varrho_1 \left[\sum_{m \in \mathbb{Z}} c_{mN} \varrho_2^{mN} + \sum_{m \in \mathbb{Z}, m \neq 0} d_{mN} \varrho_2^{mN} \right] \right\}, \quad (5.8)$$

$$\nu_0 = \frac{1}{\log \varrho_2 / \varrho_1} \left\{ \sum_{m \in \mathbb{Z}} c_{mN} \varrho_2^{mN} + \sum_{m \in \mathbb{Z}, m \neq 0} d_{mN} \varrho_2^{mN} - \left[\sum_{m \in \mathbb{Z}} c_{mN} \varrho_1^{mN} + \sum_{m \in \mathbb{Z}, m \neq 0} d_{mN} \varrho_1^{mN} \right] \right\} + d_0. \quad (5.9)$$

Next we bound the coefficients μ_k , $k = -\frac{N}{2} + 1, \dots, \frac{N}{2}$. We consider three cases: Case I: $k > 0$, Case II: $k < 0$ and Case III: $k = 0$.

Case I. $k > 0$. First

$$\begin{aligned} \left| \sum_{m \in \mathbb{Z}} c_{k+mN} \varrho_1^{k+mN} \right| &\leq \sum_{m \geq 0} |c_{k+mN}| \varrho_1^{k+mN} + \sum_{m < 0} |c_{k+mN}| \varrho_1^{k+mN} \\ &\leq \sum_{m \geq 0} M_\beta (\varrho_2 + \beta)^{-(k+mN)} \varrho_1^{k+mN} + \sum_{m < 0} M_\beta (\varrho_1 - \beta)^{-(k+mN)} \varrho_1^{k+mN} \end{aligned}$$

from (5.3) and (5.4), and by taking $\gamma_1 = \max \left\{ \frac{\varrho_1}{\varrho_2 + \beta}, \frac{\varrho_1 - \beta}{\varrho_1} \right\} < 1$, we get

$$\left| \sum_{m \in \mathbb{Z}} c_{k+mN} \varrho_1^{k+mN} \right| \leq M_\beta \gamma_1^k \sum_{m \geq 0} \gamma_1^{mN} + M_\beta \gamma_1^{N-k} \sum_{m \geq 0} \gamma_1^{Nm}$$

by observing that $k \leq N - k$, since $k \leq N/2$, we finally have

$$\left| \sum_{m \in \mathbb{Z}} c_{k+mN} \varrho_1^{k+mN} \right| \leq 2M_\beta \gamma_1^k \sum_{m \geq 0} \gamma_1^{mN} = 2M_\beta \gamma_1^k (1 - \gamma_1^N)^{-1} \leq M_1 \gamma_1^k,$$

where $M_1 = \frac{2M_\beta}{1 - \gamma_1}$. Following an identical argument, we get

$$\left| \sum_{\substack{m \in \mathbb{Z} \\ k \neq 0}} d_{k+mN} \varrho_1^{k+mN} \right| \leq M_1 \gamma_1^k.$$

Similarly, taking $\gamma_2 = \max \left\{ \frac{\varrho_2}{\varrho_2 + \beta}, \frac{\varrho_1 - \beta}{\varrho_2} \right\} < 1$, and using the fact that $\gamma_2^k < \gamma_2^{N-k}$, we obtain

$$\left| \sum_{m \in \mathbb{Z}} c_{k+mN} \varrho_2^{k+mN} \right|, \left| \sum_{\substack{m \in \mathbb{Z} \\ k \neq 0}} d_{k+mN} \varrho_2^{k+mN} \right| \leq M_2 \gamma_2^k,$$

where $M_2 = 2M_\beta \gamma_2^k (1 - \gamma_2)^{-1}$.

Case II. $k < 0$. Following arguments similar to those used in Case I, we obtain

$$\left| \sum_{m \in \mathbb{Z}} c_{k+mN} \varrho_i^{k+mN} \right|, \left| \sum_{m \in \mathbb{Z}} d_{k+mN} \varrho_i^{k+mN} \right| \leq M_i \gamma_i^{-k}, \quad i = 1, 2$$

Case III. $k = 0$. Finally, following similar arguments to cases I and II, we obtain

$$\left| \sum_{m \in \mathbb{Z}} c_{mN} \varrho_1^{mN} \right|, \left| \sum_{m \in \mathbb{Z}} d_{-mN} \varrho_1^{-mN} \right| \leq M_i, \quad i = 1, 2$$

We define γ_3 such that $\gamma_1, \gamma_2 \leq \gamma_3 = \max \left\{ \frac{\varrho_2}{\varrho_2 + \beta}, \frac{\varrho_1 - \beta}{\varrho_1} \right\} < 1$, and we take $M_3 = \max \{M_1, M_2\}$. Therefore, for $k \in \mathbb{Z}$, we have

$$|\mu_k| \leq \frac{2M_3 \gamma_3^{|k|} (\varrho_2^{-k} + \varrho_1^{-k})}{|\delta_k|}.$$

In particular, when $k > 0$ we have that

$$|\mu_k| \leq \frac{2M_3 \gamma_3^k}{\varrho_2^k} \left(1 - \left(\frac{\varrho_1}{\varrho_2} \right)^k \right)^{-1} \leq \frac{2M_3 \gamma_3^k}{\varrho_2^k} \left(1 - \frac{\varrho_1}{\varrho_2} \right)^{-1} \leq \frac{M_4 \gamma_3^k}{\varrho_2^k},$$

where $M_4 = 2M_3 \left(1 - \frac{\varrho_1}{\varrho_2} \right)^{-1}$. When $k < 0$,

$$|\mu_k| \leq M_4 \gamma_3^{-k} \varrho_1^{-k}, \quad (5.10)$$

and, when $k = 0$, we have that

$$\begin{aligned} |\mu_0| &< \frac{\log \varrho_2 (2M_3 + M_3 \log \varrho_1) + \log \varrho_1 (2M_3 + M_3 \log \varrho_2)}{\log (\varrho_2 / \varrho_1)} \\ &\leq \frac{2M_3 (\log \varrho_2 + \log \varrho_1 + \log \varrho_2 \log \varrho_1)}{\log (\varrho_2 / \varrho_1)}. \end{aligned}$$

Similarly, for the coefficients ν_k , $k > 0$, we have that $|\nu_k| \leq M_4 \gamma_3^k \varrho_1^k$, for $k < 0$, $|\nu_k| \leq M_4 \gamma_3^{-k} \varrho_2^k$, and for $k = 0$, $|\nu_0| \leq \frac{M_3 (4 + \log \varrho_2 + \log \varrho_1)}{\log (\varrho_2 / \varrho_1)}$. If we take

$$M = \max \left\{ M_4, \frac{2M_3 (\log \varrho_2 + \log \varrho_1 + \log \varrho_2 \log \varrho_1)}{\log (\varrho_2 / \varrho_1)}, \frac{M_3 (4 + \log \varrho_2 + \log \varrho_1)}{\log (\varrho_2 / \varrho_1)} \right\},$$

then

$$|\mu_k| \leq \begin{cases} M\gamma_3^k \varrho_2^{-k}, & k > 0, \\ M\gamma_3^{-k} \varrho_1^{-k}, & k < 0, \\ M, & k = 0, \end{cases} \quad |\nu_k| \leq \begin{cases} M\gamma_3^k \varrho_1^k, & k > 0, \\ M\gamma_3^{-k} \varrho_2^k, & k < 0, \\ M, & k = 0. \end{cases}$$

This leads to the following theorem with a uniform bound.

Lemma 5.1. *The coefficients μ_k, ν_k in (5.5) are bounded by:*

$$|\mu_k|, |\nu_k| \leq M\gamma_4^{|k|}, \quad (5.11)$$

where $\gamma_4 = \max\left\{\gamma_3\varrho_1, \frac{\gamma_3}{\varrho_2}\right\}$.

□

5.2. The error bound. Let us denote by $u(\cdot; \mathbf{h})$ the solution of the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = h_1 & \text{on } \partial\Omega_1, \\ u = h_2 & \text{on } \partial\Omega_2, \end{cases}$$

where $\mathbf{h} = (h_1, h_2)^T$ and by $u_N(\cdot; \mathbf{h})$ its MFS approximation. Note that u_N depends also on R_1, R_2 . Since both $u(\cdot; \mathbf{h})$ and $u_N(\cdot; \mathbf{h})$ depend linearly on h_1 and h_2 we have, with $\mathbf{f}_N = (f_1^N, f_2^N)$,

$$\begin{aligned} \|u_N(\cdot; \mathbf{f}) - u(\cdot; \mathbf{f})\|_\infty &\leq \|u(\cdot; \mathbf{f} - \mathbf{f}_N)\|_\infty + \|u_N(\cdot; \mathbf{f} - \mathbf{f}_N)\|_\infty + \|u_N(\cdot; \mathbf{f}_N) - u(\cdot; \mathbf{f}_N)\|_\infty \\ &= \|u(\cdot; \mathbf{f} - \mathbf{f}_N)\|_\infty + \|u_N(\cdot; \mathbf{f}_N) - u(\cdot; \mathbf{f}_N)\|_\infty, \end{aligned} \quad (5.12)$$

since \mathbf{f} and \mathbf{f}_N agree at the boundary points. In the following we show that, each term on the right hand side of (5.12) decays exponentially fast as $N \rightarrow \infty$.

5.2.1. The term $\|u(\cdot, \mathbf{f} - \mathbf{f}_N)\|_\infty$. From (5.2) and (5.5), we have

$$\begin{aligned} u - \tilde{u}_N &= \sum_{k \leq -\frac{N}{2}} (c_k r^k e^{ik\vartheta} + d_k r^k e^{-ik\vartheta}) + \sum_{k > \frac{N}{2}} (c_k r^k e^{ik\vartheta} + d_k r^k e^{-ik\vartheta}) \\ &\quad + \sum_{-\frac{N}{2}+1 \leq k \leq \frac{N}{2}} (\mu_k - c_k) r^k e^{ik\vartheta} + \sum_{-\frac{N}{2}+1 \leq k < 0} (d_k - \nu_k) r^{-k} e^{ik\vartheta} \\ &\quad + \sum_{0 < k \leq \frac{N}{2}} (d_k - \nu_k) r^{-k} e^{ik\vartheta} + (d_0 - \nu_0) \log r. \end{aligned} \quad (5.13)$$

By observing that the term corresponding to $m = 0$ in (5.6) vanishes, i.e.

$$\frac{\varrho_2^{-k} (c_k \varrho_1^k + d_{-k} \varrho_1^{-k}) - \varrho_1^{-k} (c_k \varrho_2^k + d_{-k} \varrho_2^{-k})}{\delta_k} - c_k = 0,$$

we have that (5.6) reduces to

$$\mu_k - c_k = \frac{1}{\delta_k} \left\{ \varrho_2^{-k} \sum_{m \in \mathbb{Z}, m \neq 0} (c_{k+mN} \varrho_1^{k+mN} + d_{-k-mN} \varrho_1^{-k-mN}) - \varrho_1^{-k} \sum_{m \in \mathbb{Z}, m \neq 0} (c_{k+mN} \varrho_2^{k+mN} + d_{-k-mN} \varrho_2^{-k-mN}) \right\}.$$

Thus,

$$\begin{aligned}
 |\mu_k - c_k| &\leq \frac{\varrho_2^{-k}}{|\delta_k|} \left\{ \sum_{m>0} |c_{k+mN}| \varrho_1^{k+mN} + \sum_{m>0} |d_{-k-mN}| \varrho_1^{-k-mN} + \sum_{m<0} |c_{k+mN}| \varrho_1^{k+mN} \right. \\
 &\quad \left. + \sum_{m<0} |d_{-k-mN}| \varrho_1^{-k-mN} \right\} \\
 &\quad + \frac{\varrho_1^{-k}}{|\delta_k|} \left\{ \sum_{m>0} |c_{k+mN}| \varrho_2^{k+mN} + \sum_{m>0} |d_{-k-mN}| \varrho_2^{-k-mN} + \sum_{m<0} |c_{k+mN}| \varrho_2^{k+mN} \right. \\
 &\quad \left. + \sum_{m<0} |d_{-k-mN}| \varrho_2^{-k-mN} \right\} \\
 &\leq \frac{\varrho_2^{-k}}{|\delta_k|} \left\{ \sum_{m>0} M_\beta (\varrho_2 + \beta)^{-(k+mN)} \varrho_1^{k+mN} + \sum_{m<0} M_\beta (\varrho_1 - \beta)^{-(k+mN)} \varrho_1^{k+mN} \right. \\
 &\quad \left. + \sum_{m>0} M_\beta (\varrho_1 - \beta)^{k+mN} \varrho_1^{-(k+mN)} + \sum_{m<0} M_\beta (\varrho_2 + \beta)^{k+mN} \varrho_1^{-(k+mN)} \right\} \\
 &\quad + \frac{\varrho_1^{-k}}{|\delta_k|} \left\{ \sum_{m>0} M_\beta (\varrho_2 + \beta)^{-(k+mN)} \varrho_2^{k+mN} + \sum_{m<0} M_\beta (\varrho_1 - \beta)^{-(k+mN)} \varrho_2^{k+mN} \right. \\
 &\quad \left. + \sum_{m>0} M_\beta (\varrho_1 - \beta)^{k+mN} \varrho_2^{-(k+mN)} + \sum_{m<0} M_\beta (\varrho_2 + \beta)^{k+mN} \varrho_2^{-(k+mN)} \right\} \\
 &\leq \frac{M_\beta \varrho_2^{-k}}{|\delta_k|} \left\{ \gamma_3^{k+N} \sum_{m=0}^{\infty} \gamma_3^{Nm} + \gamma_3^{N-k} \sum_{m=0}^{\infty} \gamma_3^{Nm} + \gamma_3^{N+k} \sum_{m=0}^{\infty} \gamma_3^{Nm} + \gamma_3^{N-k} \sum_{m=0}^{\infty} \gamma_3^{Nm} \right\} \\
 &\quad + \frac{M_\beta \varrho_1^{-k}}{|\delta_k|} \left\{ \gamma_3^{k+N} \sum_{m=0}^{\infty} \gamma_3^{Nm} + \gamma_3^{N-k} \sum_{m=0}^{\infty} \gamma_3^{Nm} + \gamma_3^{N+k} \sum_{m=0}^{\infty} \gamma_3^{Nm} + \gamma_3^{N-k} \sum_{m=0}^{\infty} \gamma_3^{Nm} \right\},
 \end{aligned}$$

If $k > 0$, then $k + N > N - k$, therefore

$$|\mu_k - c_k| \leq \frac{8M_\beta \varrho_2^k \gamma_3^{N-k}}{\varrho_2^{2k} - \varrho_1^{2k}} \sum_{m=0}^{\infty} \gamma_3^{Nm} \leq \frac{8M_\beta \gamma_3^{N-k}}{\varrho_2^k \left(1 - \frac{\varrho_2^2}{\varrho_1^2}\right) (1 - \gamma_3)} \leq \frac{M_5 \gamma_3^{N-k}}{\varrho_2^k}$$

where $M_5 = \frac{8M_\beta}{(1-\gamma_3)} \left(1 - \frac{\varrho_2^2}{\varrho_1^2}\right)^{-1}$. If $k < 0$ then $k + N < N - k$, therefore

$$\begin{aligned}
 |\mu_k - c_k| &\leq \frac{4M_\beta \varrho_2^{-k} \gamma_3^{k+N}}{|\delta_k|} \sum_{m=0}^{\infty} \gamma_3^{Nm} + \frac{4M_\beta \varrho_1^{-k} \gamma_3^{k+N}}{|\delta_k|} \sum_{m=0}^{\infty} \gamma_3^{Nm} \\
 &\leq \frac{8M_\beta \varrho_1^{-k} \gamma_3^{k+N}}{\left(1 - \frac{\varrho_2^2}{\varrho_1^2}\right) (1 - \gamma_3)} \leq M_5 \gamma_3^{N+k} \varrho_1^{-k}.
 \end{aligned}$$

When $k = 0$, from (5.8) and by observing that

$$\frac{\log \varrho_2 c_0 - \log \varrho_1 c_0}{\log \varrho_2 / \varrho_1} - c_0 = 0,$$

we have

$$\begin{aligned}
 \mu_0 - c_0 &= \frac{1}{\log \varrho_2 / \varrho_1} \left\{ \log \varrho_2 \left[\sum_{m \in \mathbb{Z}} c_{mN} \varrho_1^{mN} + \sum_{m \in \mathbb{Z}, m \neq 0} d_{-mN} \varrho_1^{-mN} + d_0 \log \varrho_1 \right] \right. \\
 &\quad \left. - \log \varrho_1 \left[\sum_{m \in \mathbb{Z}} c_{mN} \varrho_2^{mN} + \sum_{m \in \mathbb{Z}, m \neq 0} d_{-mN} \varrho_2^{-mN} + d_0 \log \varrho_2 \right] \right\} - c_0.
 \end{aligned}$$

Thus,

$$\begin{aligned}
|\mu_0 - c_0| &= \frac{1}{\log \varrho_2 / \varrho_1} \left| \log \varrho_2 \sum_{m \neq 0} (c_{mN} \varrho_1^{mN} + d_{-mN} \varrho_1^{-mN}) \right. \\
&\quad \left. - \log \varrho_1 \sum_{m \neq 0} (c_{mN} \varrho_2^{mN} + d_{-mN} \varrho_2^{-mN}) \right| \\
&= \frac{1}{\log \varrho_2 / \varrho_1} \left| \left\{ \log \varrho_2 \left[\sum_{m > 0} (c_{mN} \varrho_1^{mN} + d_{-mN} \varrho_1^{-mN}) + \sum_{m < 0} (c_{mN} \varrho_1^{mN} + d_{-mN} \varrho_1^{-mN}) \right] \right. \right. \\
&\quad \left. \left. + \log \varrho_1 \left[\sum_{m > 0} (c_{mN} \varrho_2^{mN} + d_{-mN} \varrho_2^{-mN}) + \sum_{m < 0} (c_{mN} \varrho_2^{mN} + d_{-mN} \varrho_2^{-mN}) \right] \right\} \right| \\
&\leq \frac{\log \varrho_2}{\log \varrho_2 / \varrho_1} \left\{ \sum_{m > 0} M_\beta (\varrho_2 + \beta)^{-mN} \varrho_1^{mN} + \sum_{m < 0} M_\beta (\varrho_1 - \beta)^{-mN} \varrho_1^{mN} \right. \\
&\quad \left. + \sum_{m > 0} M_\beta (\varrho_1 - \beta)^{mN} \varrho_1^{-mN} + \sum_{m < 0} M_\beta (\varrho_2 + \beta)^{mN} \varrho_1^{-mN} \right\} \\
&\quad + \frac{1}{\log \varrho_2 / \varrho_1} \left\{ \sum_{m > 0} M_\beta (\varrho_2 + \beta)^{-mN} \varrho_2^{mN} + \sum_{m < 0} M_\beta (\varrho_1 - \beta)^{-mN} \varrho_2^{mN} \right. \\
&\quad \left. + \sum_{m > 0} M_\beta (\varrho_1 - \beta)^{mN} \varrho_2^{-mN} + \sum_{m < 0} M_\beta (\varrho_2 + \beta)^{mN} \varrho_2^{-mN} \right\} \\
&\leq \frac{8M_\beta \log \varrho_2}{\log \varrho_2 / \varrho_1} \gamma_3^N \sum_{m=0}^{\infty} \gamma_3^{Nm} \leq \frac{8M_\beta \log \varrho_2}{\log \varrho_2 / \varrho_1} \frac{\gamma_3^N}{1 - \gamma_3} = M_6 \gamma_3^N,
\end{aligned}$$

where $M_6 = \frac{8M_\beta \log \varrho_2}{(1 - \gamma_3) \log \varrho_2 / \varrho_1}$. Following similar arguments we have that

$$|\nu_k - d_k| \leq \begin{cases} M_5 \gamma_3^{N-k} \varrho_1^k, & k > 0 \\ M_5 \gamma_3^{N+k} \varrho_2^k, & k < 0 \\ M_6 \gamma_3^N & k = 0. \end{cases}$$

If we take $M_7 = \max \{M_5, M_6\}$ we obtain the lemma

Lemma 5.2. *The terms $\mu_k - c_k$ and $\nu_k - d_k$ in equation (5.13) are bounded by:*

$$|\mu_k - c_k| \leq \begin{cases} M_7 \gamma_3^{N-k} \varrho_2^{-k}, & k > 0, \\ M_7 \gamma_3^{N+k} \varrho_1^{-k}, & k < 0, \\ M_7 \gamma_3^N, & k = 0, \end{cases} \quad |\nu_k - d_k| \leq \begin{cases} M_7 \gamma_3^{N-k} \varrho_1^k, & k > 0, \\ M_7 \gamma_3^{N+k} \varrho_2^k, & k < 0, \\ M_7 \gamma_3^N, & k = 0. \end{cases}$$

□

We are now in a position to bound

$$\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} (\mu_k - c_k) r^k e^{ik\vartheta} + \sum_{k=-\frac{N}{2}+1}^{-1} (\nu_k - d_k) r^{-k} e^{ik\vartheta} + \sum_{k=1}^{\frac{N}{2}} (\nu_k - d_k) r^{-k} e^{ik\vartheta} + (\nu_0 - d_0) \log r.$$

In the case when the error attains its maximum value on the inner circle $r = \varrho_1$, we have, using Lemma 5.2,

$$\begin{aligned}
 & \left| \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} (\mu_k - c_k) \varrho_1^k e^{ik\vartheta} + \sum_{k=-\frac{N}{2}+1}^{-1} (\nu_k - d_k) \varrho_1^{-k} e^{ik\vartheta} + \sum_{k=1}^{\frac{N}{2}} (\nu_k - d_k) \varrho_1^{-k} e^{ik\vartheta} + (\nu_0 - d_0) \log \varrho_1 \right| \\
 & \leq \sum_{k=1}^{\frac{N}{2}} |\mu_k - c_k| \varrho_1^k + \sum_{k=-\frac{N}{2}+1}^{-1} |\mu_k - c_k| \varrho_1^k + |\mu_0 - c_0| \\
 & \quad + \sum_{k=1}^{\frac{N}{2}} |\nu_k - d_k| \varrho_1^{-k} + \sum_{k=-\frac{N}{2}+1}^{-1} |\nu_k - d_k| \varrho_1^{-k} + |\nu_0 - d_0| \log \varrho_1 \\
 & \leq M_8 \left\{ \sum_{k=1}^{\frac{N}{2}} \gamma_3^{N-k} \left(\frac{\varrho_1}{\varrho_2} \right)^k + \sum_{k=-\frac{N}{2}+1}^{-1} \gamma_3^{N+k} \varrho_1^{-k} \varrho_1^k + \gamma_3^N + \sum_{k=1}^{\frac{N}{2}} \gamma_3^{N-k} \varrho_1^k \varrho_1^{-k} + \sum_{k=-\frac{N}{2}+1}^{-1} \gamma_3^{N+k} \left(\frac{\varrho_1}{\varrho_2} \right)^{-k} + \gamma_3^N \right\} \\
 & \leq M_8 \left\{ \sum_{k=1}^{\frac{N}{2}} \gamma_3^{N-k} + \sum_{k=1}^{\frac{N}{2}} \gamma_3^{N-k} + \gamma_3^N + \sum_{k=1}^{\frac{N}{2}} \gamma_3^{N-k} + \sum_{k=1}^{\frac{N}{2}} \gamma_3^{N-k} + \gamma_3^N \right\} \\
 & \leq 4M_8 \left\{ \gamma_3^{\frac{N}{2}} + \gamma_3^N \right\} \leq 8M_8 \gamma_3^{\frac{N}{2}}, \tag{5.14}
 \end{aligned}$$

where $M_8 = \max \{M_7, \log \varrho_1, \log \varrho_2\}$. Similarly, when the maximum is attained on the outer circle $r = \varrho_2$, we have

$$\begin{aligned}
 & \left| \sum_{k=-\frac{N}{2}+1}^{-1} (c_k - \mu_k) \varrho_2^k e^{ik\vartheta} + \sum_{k=-\frac{N}{2}+1}^{-1} (d_k - \nu_k) \varrho_2^{-k} e^{ik\vartheta} + \sum_{k=1}^{\frac{N}{2}} (d_k - \nu_k) \varrho_2^{-k} e^{ik\vartheta} + (d_0 - \nu_0) \log \varrho_2 \right| \\
 & \leq 4M_8 \left\{ \gamma_3^{\frac{N}{2}} + \gamma_3^N \right\} \leq 8M_8 \gamma_3^{\frac{N}{2}}. \tag{5.15}
 \end{aligned}$$

The remaining term in (5.13) can be bounded in the following way:

$$\begin{aligned}
 & \left| \sum_{k > \frac{N}{2}} (c_k r^k e^{ik\vartheta} + d_k r^k e^{-ik\vartheta}) + \sum_{k \leq \frac{N}{2}} (c_k r^k e^{ik\vartheta} + d_k r^k e^{-ik\vartheta}) \right| \\
 & \leq \sum_{k > \frac{N}{2}} |c_k| r^k + \sum_{k > \frac{N}{2}} |d_k| r^k + \sum_{k \leq -\frac{N}{2}} |c_k| r^k + \sum_{k \leq -\frac{N}{2}} |d_k| r^k \\
 & \leq \sum_{k=\frac{N}{2}+1}^{\infty} M_\beta \left(\frac{r}{\varrho_2 + \beta} \right)^k + \sum_{k=\frac{N}{2}+1}^{\infty} M_\beta \left(\frac{r}{\varrho_2 + \beta} \right)^k + \sum_{k \leq -\frac{N}{2}} M_\beta \left(\frac{r}{\varrho_1 - \beta} \right)^k + \sum_{k \leq -\frac{N}{2}} M_\beta \left(\frac{r}{\varrho_1 - \beta} \right)^k \\
 & \leq 2M_\beta \left\{ \sum_{k=\frac{N}{2}+1}^{\infty} \left(\frac{r}{\varrho_2 + \beta} \right)^k + \sum_{k=\frac{N}{2}}^{\infty} \left(\frac{\varrho_1 - \beta}{r} \right)^k \right\} \\
 & \leq 2M_\beta \left\{ \sum_{k=\frac{N}{2}}^{\infty} \left(\frac{r}{\varrho_2 + \beta} \right)^k + \sum_{k=\frac{N}{2}}^{\infty} \left(\frac{\varrho_1 - \beta}{r} \right)^k \right\} \\
 & = 2M_\beta \left\{ \left(\frac{r}{\varrho_2 + \beta} \right)^{\frac{N}{2}} \sum_{k=0}^{\infty} \left(\frac{r}{\varrho_2 + \beta} \right)^k + \left(\frac{\varrho_1 - \beta}{r} \right)^{\frac{N}{2}} \sum_{k=0}^{\infty} \left(\frac{\varrho_1 - \beta}{r} \right)^k \right\} \\
 & = 2M_\beta \left\{ \left(\frac{r}{\varrho_2 + \beta} \right)^{\frac{N}{2}} \left(1 - \frac{r}{\varrho_2 + \beta} \right)^{-1} + \left(\frac{\varrho_1 - \beta}{r} \right)^{\frac{N}{2}} \left(1 - \frac{\varrho_1 - \beta}{r} \right)^{-1} \right\}.
 \end{aligned}$$

Therefore, when the maximum is attained on the inner circle $r = \varrho_1$, we have

$$\begin{aligned} & \left| \sum_{k > \frac{N}{2}} (c_k \varrho_1^k e^{ik\vartheta} + d_k \varrho_1^k e^{-ik\vartheta}) + \sum_{k \leq \frac{N}{2}} (c_k \varrho_1^k e^{ik\vartheta} + d_k \varrho_1^k e^{-ik\vartheta}) \right| \\ & \leq 2M_\beta \left\{ \left(1 - \frac{\varrho_1}{\varrho_2 + \beta}\right)^{-1} + \left(1 - \frac{\varrho_1 - \beta}{\varrho_1}\right)^{-1} \right\} \gamma_3^{\frac{N}{2}} = M_1 \gamma_3^{\frac{N}{2}} \leq M_8 \gamma_3^{\frac{N}{2}}. \end{aligned} \quad (5.16)$$

Similarly, when the maximum is obtained on the external circle $r = \varrho_2$, we have

$$\begin{aligned} & \left| \sum_{k > \frac{N}{2}} (c_k \varrho_2^k e^{ik\vartheta} + d_k \varrho_2^k e^{-ik\vartheta}) + \sum_{k \leq \frac{N}{2}} (c_k \varrho_2^k e^{ik\vartheta} + d_k \varrho_2^k e^{-ik\vartheta}) \right| \\ & \leq 2M_1 \left\{ \left(1 - \frac{\varrho_2}{\varrho_2 + \beta}\right)^{-1} + \left(1 - \frac{\varrho_1 - \beta}{\varrho_2}\right)^{-1} \right\} \gamma_3^{\frac{N}{2}} = M_2 \gamma_3^{\frac{N}{2}} \leq M_8 \gamma_3^{\frac{N}{2}}. \end{aligned} \quad (5.17)$$

Therefore, we have the following theorem:

Theorem 5.3.

$$\|u_N(\cdot, \mathbf{f} - \mathbf{f}_N)\|_\infty \leq 9M_8 \gamma_3^{\frac{N}{2}}. \quad (5.18)$$

□

5.2.2. *The term $\|u_N(\cdot; \mathbf{f}_N) - u(\cdot; \mathbf{f}_N)\|_\infty$.* To bound this term, we need to investigate how well the MFS approximates problems corresponding to exact solutions of the form $z^k, k = \frac{-N}{2}, \dots, \frac{N}{2}$, and $\log|z|$ which in polar coordinates take the form $r^k e^{ik\varphi}, \log r$, $r \in [\varrho_1, \varrho_2]$ and $\varphi \in [0, 2\pi]$. For simplicity we denote this exact solution by $u(\cdot; r^k e^{ik\varphi})$ and its MFS approximation by $u_N(\cdot; r^k e^{ik\varphi})$. Thus we need to bound

$$\begin{aligned} & \|u_N(\cdot; r^k e^{ik\varphi}) - u(\cdot; r^k e^{ik\varphi})\|_\infty, \quad k = -N/2 + 1, \dots, N/2, \quad \text{and} \\ & \|u_N(\cdot; \log r) - u(\cdot; \log r)\|_\infty. \end{aligned}$$

Without loss of generality, we assume that $\varrho_1 = 1$. We shall prove the following lemma:

Lemma 5.4. *We have the following estimates:*

$$\|u_N(\cdot; r^k e^{ik\varphi}) - u(\cdot; r^k e^{ik\varphi})\|_\infty \leq CN^4 \omega^{N-2|k|}, \quad (5.19)$$

where $k = -N/2 + 1, \dots, N/2$, $\omega = \max\left\{R_1, \frac{1}{R_2}\right\} < 1$ and C is a constant independent of k and N .

Proof. Since u and u_N are harmonic in the annulus, from the maximum principle

$$\|u_N - u\|_\infty = \max\left\{\sup_{r=\varrho_1} |u_N - u|, \sup_{r=\varrho_2} |u_N - u|\right\}.$$

First, we bound the term $u_N(\cdot; r^k e^{ik\varphi}) - u(\cdot; r^k e^{ik\varphi})$ on the circle $r = \varrho_1 = 1$.

Case I. $k = 0$.

In this case, the exact solution is $u \equiv 1$, i.e., $f_1, f_2 \equiv 1$, and thus $\mathbf{f}_1 = \mathbf{f}_2 = (1, 1, \dots, 1)$, and

$$\langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle = \begin{cases} \sqrt{N} & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle = \begin{cases} \sqrt{N} & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, from expression (4.2), we have

$$u_N(\cdot; 1) = \frac{\sqrt{N}}{\lambda_1^{11} \lambda_1^{22} - \lambda_1^{12} \lambda_1^{21}} \{(\lambda_1^{11} - \lambda_1^{21}) \langle \mathbf{l}_2, \boldsymbol{\xi}_1 \rangle + (\lambda_1^{22} - \lambda_1^{12}) \langle \mathbf{l}_1, \boldsymbol{\xi}_1 \rangle\},$$

where

$$\begin{aligned}\lambda_1^{11} &= \frac{N}{2\pi} \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN}, & \lambda_1^{12} &= \frac{N}{2\pi} \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} - \log R_2 \right\}, \\ \lambda_1^{21} &= \frac{N}{2\pi} \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} - \log \varrho_2 \right\}, & \lambda_1^{22} &= \frac{N}{2\pi} \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} - \log R_2 \right\},\end{aligned}$$

and

$$\langle \overline{\mathbf{l}_1}, \overline{\boldsymbol{\xi}_1} \rangle = \frac{\sqrt{N}}{2\pi} \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \cos(mN\vartheta), \quad \langle \overline{\mathbf{l}_2}, \overline{\boldsymbol{\xi}_1} \rangle = \frac{\sqrt{N}}{2\pi} \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right\}.$$

Therefore,

$$\|u_N(\cdot, \cdot; 1) - u(\cdot; \cdot, \cdot; 1)\|_{\infty} = \max_{\vartheta \in [0, 2\pi]} |u_N(\cos \vartheta, \sin \vartheta; 1) - 1| = \left| \frac{A - B}{B} \right|$$

where

$$\begin{aligned}A &= \frac{N}{2\pi} \left\{ (\lambda_1^{11} - \lambda_1^{21}) \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right\} \right. \\ &\quad \left. + (\lambda_1^{22} - \lambda_1^{12}) \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \cos(mN\vartheta) \right\},\end{aligned}$$

and $B = \lambda_1^{11} \lambda_1^{22} - \lambda_1^{12} \lambda_1^{21}$. We thus have

$$\begin{aligned}|B| &= \frac{1}{4\pi^2} \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \left\{ N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \right\} \\ &\quad + \left\{ N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{R_2} \right)^{mN} \right\} \left\{ N \log \varrho_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right\} \geq N^2 \log R_2 \log \varrho_2,\end{aligned}$$

and

$$\begin{aligned}|A - B| &= \left| \frac{N}{2\pi} \left\{ (\lambda_1^{11} - \lambda_1^{21}) \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right\} \right. \right. \\ &\quad \left. \left. + (\lambda_1^{22} - \lambda_1^{12}) \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \cos(mN\vartheta) \right\} - (\lambda_1^{11} \lambda_1^{22} - \lambda_1^{12} \lambda_1^{21}) \right| \\ &= \left| \frac{N}{2\pi} \left\{ \left(\frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} + \frac{1}{2\pi} \left[N \log \varrho_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right] \right) \right. \right. \\ &\quad \times \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right) \\ &\quad \left. \left. + \left(\frac{1}{2\pi} \left[\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} - N \log R_2 \right] - \frac{1}{2\pi} \left[\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{R_2} \right)^{mN} - N \log R_2 \right] \right) \right. \right. \\ &\quad \left. \left. \times \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \cos(mN\vartheta) \right\} \right| \\ &\quad + \frac{1}{4\pi^2} \left\{ \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \left[N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \right] \right. \\ &\quad \left. + \left[N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{R_2} \right)^{mN} \right] \left[N \log \varrho_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right] \right\} | \end{aligned}$$

$$\begin{aligned}
&= \left| \frac{N}{4\pi^2} \left\{ \left(\sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} + N \log \varrho_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right) \left(\sum_{m=1}^{\infty} \frac{1}{mN} \frac{1}{R_2^{mN}} \cos(mN\vartheta) \right) \right. \right. \\
&\quad \left. \left. + \left(\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{R_2^{mN}} \right) \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \cos(mN\vartheta) \right\} \right. \\
&\quad \left. + \frac{1}{4\pi^2} \left\{ \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \left[N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \right] \right. \right.
\end{aligned}$$

$$\begin{aligned}
 & + \left[N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{R_2} \right)^{mN} \right] \left[N \log \varrho_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right] \Big| \\
 = & \frac{1}{4\pi^2} \left| \left\{ N \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \sum_{m=1}^{\infty} \frac{1}{mN} \frac{1}{R_2^{mN}} \cos(mN\vartheta) - N \log R_2 \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \right. \right. \\
 & + N^2 \log \varrho_2 \sum_{m=1}^{\infty} \frac{1}{mN} \frac{1}{R_2^{mN}} \cos(mN\vartheta) - N^2 \log \varrho_2 \log R_2 \\
 & - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{R_2^{mN}} \cos(mN\vartheta) + N \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \log R_2 \\
 & + N \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \cos(mN\vartheta) \\
 & - N \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{R_2^{mN}} \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \cos(mN\vartheta) \\
 & + N \log R_2 \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} - \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \\
 & + N^2 \log \varrho_2 \log R_2 - N \log R_2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \\
 & \left. - N \log \varrho_2 \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{R_2^{mN}} + \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{R_2^{mN}} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right\} \Big| \\
 \leq & \sum_{m=1}^{\infty} R_1^{mN} \sum_{m=1}^{\infty} \frac{1}{R_2^{mN}} + N \log R_2 \sum_{m=1}^{\infty} R_1^{mN} + N^2 \log \varrho_2 \sum_{m=1}^{\infty} \frac{1}{R_2^{mN}} \\
 & + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{mN} \sum_{m=1}^{\infty} \frac{1}{R_2^{mN}} + N \log R_2 \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{mN} \\
 & + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{mN} \sum_{m=1}^{\infty} R_1^{mN} + \sum_{m=1}^{\infty} \frac{1}{R_2^{mN}} \sum_{m=1}^{\infty} R_1^{mN} + \\
 & + N \log R_2 \sum_{m=1}^{\infty} R_1^{mN} + \sum_{m=1}^{\infty} R_1^{mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{mN} + N \log R_2 \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{mN} \\
 & + N \log \varrho_2 \sum_{m=1}^{\infty} \frac{1}{R_2^{mN}} + \sum_{m=1}^{\infty} \frac{1}{R_2^{mN}} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{mN} \\
 \leq & 2R_1^N \frac{2}{R_2^N} + N \log R_2 2R_1^N + N^2 \log \varrho_2 \frac{2}{R_2^N} + 2 \left(\frac{R_1}{\varrho_2} \right)^N \frac{2}{R_2^N} \\
 & + N \log R_2 2 \left(\frac{R_1}{\varrho_2} \right)^N + 4 \left(\frac{\varrho_2}{R_2} \right)^N R_1^N + \frac{2}{R_2^N} 2R_1^N \\
 & + 2N \log R_2 R_1^N + 4R_1^N \left(\frac{\varrho_2}{R_2} \right)^N + 2N \log R_2 \left(\frac{R_1}{\varrho_2} \right)^N + 2N \log \varrho_2 \frac{1}{R_2^N} + 4 \frac{1}{R_2^N} \left(\frac{R_1}{\varrho_2} \right)^N .
 \end{aligned}$$

We take $\omega = \max \left\{ R_1, \frac{\varrho_2}{R_2} \right\} < 1$ and therefore:

$$\begin{aligned} |A - B| &\leq 4\omega^{2N} + 2N \log R_2 \omega^N + 2N^2 \log \varrho_2 \omega^N + 4\omega^{2N} + 2N \log R_2 \omega^{2N} + 4\omega^{2N} \\ &\quad + 4\omega^{2N} + 2N \log R_2 \omega^N + 4\omega^{2N} + 2N \log R_2 \omega^N + 2N \log R_2 \omega^N + 4\omega^{2N} \\ &\leq 4\omega^N + 2N \log R_2 \omega^N + 2N^2 \log R_2 \omega^N + 4\omega^N + 2N \log R_2 \omega^N + 4\omega^N + 4\omega^N + 2N \log R_2 \omega^N + 4\omega^N \\ &\quad + 2N \log R_2 \omega^N + 2N \log R_2 \omega^N + 4\omega^N \leq 24\omega^N + 12N^2 \log R_2 \omega^N. \end{aligned}$$

Thus if $\log R_2 > 1$ we have:

$$\begin{aligned} \frac{|A - B|}{|B|} &\leq \frac{24\omega^N + 12N^2 \log R_2 \omega^N}{N^2 \log R_2 \log \varrho_2} \leq \frac{24N^2 \log R_2 \omega^N + 12N^2 \log R_2 \omega^N}{N^2 \log R_2 \log \varrho_2} \leq \frac{36N^2 \log R_2 \omega^N}{N^2 \log R_2 \log \varrho_2} \\ &= \frac{36\omega^N}{\log \varrho_2} \end{aligned}$$

If $\log R_2 < 1$ we have:

$$\frac{|A - B|}{|B|} \leq \frac{24\omega^N + 12N^2 \log R_2 \omega^N}{N^2 \log R_2 \log \varrho_2} \leq \frac{24N^2 \omega^N + 12N^2 \omega^N}{N^2 \log R_2 \log \varrho_2} = \frac{36\omega^N}{\log R_2 \log \varrho_2} = \frac{36\omega^N}{\log \varrho_2 \log R_2}$$

Case II, $0 < k \leq N/2$: From (4.2) we have:

$$\begin{aligned} u_N &= \sum_{j=1}^N \frac{\lambda_j^{11}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_j \rangle} - \sum_{j=1}^N \frac{\lambda_j^{21}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_j \rangle} \\ &\quad - \sum_{j=1}^N \frac{\lambda_j^{12}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_j \rangle} + \sum_{j=1}^N \frac{\lambda_j^{22}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_j \rangle}, \end{aligned}$$

where

$$\langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle = \begin{cases} \sqrt{N} & \text{if } j = k+1 \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle = \begin{cases} \sqrt{N} \varrho_2^k & \text{if } j = k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$u_N = \frac{\sqrt{N}}{\lambda_{k+1}^{11} \lambda_{k+1}^{22} - \lambda_{k+1}^{12} \lambda_{k+1}^{21}} \{ (\varrho_2^k \lambda_{k+1}^{11} - \lambda_{k+1}^{21}) \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_{k+1} \rangle} + (\lambda_{k+1}^{22} - \varrho_2^k \lambda_{k+1}^{12}) \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_{k+1} \rangle} \}$$

We can therefore write

$$\left| \frac{A}{B} - e^{ik\theta} \right| = \frac{|A - e^{ik\theta} B|}{|B|}$$

where

$$A = \sqrt{N} \{ (\varrho_2^k \lambda_{k+1}^{11} - \lambda_{k+1}^{21}) \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_{k+1} \rangle} + (\lambda_{k+1}^{22} - \varrho_2^k \lambda_{k+1}^{12}) \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_{k+1} \rangle} \}$$

and

$$B = \lambda_{k+1}^{11} \lambda_{k+1}^{22} - \lambda_{k+1}^{12} \lambda_{k+1}^{21}.$$

Since $\varrho_1 = 1$, we have

$$\begin{aligned} \lambda_{k+1}^{11} &= \frac{N}{4\pi} \sum_{m=0}^{\infty} \left\{ R_1^{k+mN} \frac{1}{k+mN} + R_1^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ \lambda_{k+1}^{12} &= \frac{N}{4\pi} \sum_{m=0}^{\infty} \left\{ \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ \lambda_{k+1}^{21} &= \frac{N}{4\pi} \sum_{m=0}^{\infty} \left\{ \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ \lambda_{k+1}^{22} &= \frac{N}{4\pi} \sum_{m=0}^{\infty} \left\{ \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \end{aligned}$$

and

$$\begin{aligned} \langle \overline{\mathbf{l}_1}, \boldsymbol{\xi}_{k+1} \rangle &= \frac{\sqrt{N}}{4\pi} \sum_{m=1}^{\infty} \left\{ R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\ \langle \overline{\mathbf{l}_2}, \boldsymbol{\xi}_{k+1} \rangle &= \frac{\sqrt{N}}{4\pi} \sum_{m=1}^{\infty} \left\{ \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\}. \end{aligned}$$

Next, we obtain a lower bound for $|B|$.

$$\begin{aligned} B &= \sum_{m=0}^{\infty} \left\{ \frac{R_1^{k+mN}}{k+mN} + \frac{R_1^{N-k+mN}}{N-k+mN} \right\} \sum_{m=0}^{\infty} \left\{ \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \right. \\ &\quad \left. + \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ &\quad - \sum_{m=0}^{\infty} \left\{ \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ &\quad \times \sum_{m=0}^{\infty} \left\{ \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ &= \left\{ \frac{R_1}{k} + \frac{R_1^{N-k}}{N-k} + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \right\} \\ &\quad \times \left\{ \frac{1}{k} \left(\frac{\varrho_2}{R_2} \right)^k + \frac{1}{N-k} \left(\frac{\varrho_2}{R_2} \right)^{N-k} \right. \\ &\quad \left. + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ &\quad - \left\{ \frac{1}{kR_2^k} + \frac{1}{(N-k)R_2^{N-k}} + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ &\quad \times \left\{ \frac{1}{k} \left(\frac{R_1}{\varrho_2} \right)^k + \frac{1}{N-k} \left(\frac{R_1}{\varrho_2} \right)^{N-k} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ &= \frac{1}{k^2} \frac{R_1^k \varrho_2^k}{R_2^k} + \frac{1}{k(N-k)} \frac{R_1^k \varrho_2^{N-k}}{R_2^{N-k}} + \frac{R_1^k}{k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \\ &\quad + \frac{R_1^k}{k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \\ &\quad + \frac{1}{k(N-k)} \frac{R_1^{N-k} \varrho_2^k}{R_2^k} + \frac{1}{(N-k)^2} \left(\frac{R_1 \varrho_2}{R_2} \right)^{N-k} + \frac{R_1^{N-k}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \\ &\quad + \frac{R_1^{N-k}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \\ &\quad + \frac{1}{k} \left(\frac{\varrho_2}{R_2} \right)^k \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} + \frac{1}{N-k} \left(\frac{\varrho_2}{R_2} \right)^{N-k} \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} \end{aligned}$$

$$\begin{aligned}
& + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \\
& + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \\
& + \frac{1}{k} \left(\frac{\varrho_2}{R_2} \right)^k \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} + \frac{1}{N-k} \left(\frac{\varrho_2}{R_2} \right)^{N-k} \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \\
& + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \\
& + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \\
& - \frac{1}{k^2} \frac{R_1^k}{\varrho_2^k R_2^k} - \frac{1}{k(N-k)} \frac{R_1^{N-k}}{\varrho_2^{N-k} R_2^k} - \frac{1}{k R_2^k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \\
& - \frac{1}{k R_2^k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \\
& - \frac{1}{k(N-k)} \frac{R_1^k}{\varrho_2^k R_2^{N-k}} - \frac{1}{(N-k)^2} \left(\frac{R_1}{\varrho_2 R_2} \right)^{N-k} - \frac{1}{(N-k) R_2^{N-k}} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \\
& - \frac{1}{(N-k) R_2^{N-k}} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \\
& - \frac{1}{k} \left(\frac{R_1}{\varrho_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} - \frac{1}{N-k} \left(\frac{R_1}{\varrho_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} \\
& - \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \\
& - \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \\
& - \frac{1}{k} \left(\frac{R_1}{\varrho_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} - \frac{1}{N-k} \left(\frac{R_1}{\varrho_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \\
& - \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \\
& - \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \\
& = \frac{1}{k^2} \frac{R_1^k}{R_2^k} \left(\varrho_2^k - \frac{1}{\varrho_2^k} \right) + \frac{1}{k(N-k)} \left(\frac{R_1^k \varrho_2^{N-k}}{R_2^{N-k}} - \frac{R_1^k}{\varrho_2^k R_2^{N-k}} \right) \\
& + \frac{1}{k(N-k)} \left(\frac{R_1^{N-k} \varrho_2^k}{R_2^k} - \frac{R_1^{N-k}}{\varrho_2^{N-k} R_2^k} \right) \\
& + \frac{1}{k} \sum_{m=1}^{\infty} \left\{ \frac{R_1^k \varrho_2^{k+mN}}{R_2^{k+mN}} - \frac{R_1^k}{\varrho_2^k R_2^{k+mN}} \right\} \frac{1}{k+mN}
\end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{k} \sum_{m=1}^{\infty} \left\{ \frac{R_1^{k+mN} \varrho_2^k}{R_2^k} - \frac{R_1^{k+mN}}{\varrho_2^{k+mN} R_2^k} \right\} \frac{1}{k+mN} \\
 & + \frac{1}{k} \sum_{m=1}^{\infty} \left\{ \frac{R_1^k \varrho_2^{N-k+mN}}{R_2^{N-k+mN}} - \frac{R_1^k}{\varrho_2^k R_2^{N-k+mN}} \right\} \frac{1}{N-k+mN} \\
 & + \frac{1}{k} \sum_{m=1}^{\infty} \left\{ \frac{R_1^{N-k+mN} \varrho_2^k}{R_2^k} - \frac{R_1^{N-k+mN}}{\varrho_2^{N-k+mN} R_2^k} \right\} \frac{1}{N-k+mN} \\
 & + \frac{1}{(N-k)^2} \left\{ \left(\frac{R_1 \varrho_2}{R_2} \right)^{N-k} - \left(\frac{R_1}{\varrho_2 R_2} \right)^{N-k} \right\} \\
 & + \frac{1}{N-k} \sum_{m=1}^{\infty} \left\{ \frac{R_1^{N-k} \varrho_2^{k+mN}}{R_2^{k+mN}} - \frac{R_1^{N-k}}{\varrho_2^{N-k} R_2^{k+mN}} \right\} \frac{1}{k+mN} \\
 & + \frac{1}{N-k} \sum_{m=1}^{\infty} \left\{ \frac{\varrho_2^{N-k} R_1^{k+mN}}{R_2^{N-k}} - \frac{R_1^{k+mN}}{\varrho_2^{k+mN} R_2^{N-k}} \right\} \frac{1}{k+mN} + \\
 & \frac{1}{N-k} \sum_{m=1}^{\infty} \left\{ \frac{R_1^{N-k} \varrho_2^{N-k+mN}}{R_2^{N-k+mN}} - \frac{R_1^{N-k}}{\varrho_2^{N-k} R_2^{N-k+mN}} \right\} \frac{1}{N-k+mN} \\
 & + \frac{1}{N-k} \sum_{m=1}^{\infty} \left\{ \frac{\varrho_2^{N-k} R_1^{N-k+mN}}{R_2^{N-k}} - \frac{R_1^{N-k+mN}}{\varrho_2^{N-k+mN} R_2^{N-k}} \right\} \frac{1}{N-k+mN} \\
 & \left\{ \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \right. \\
 & \left. - \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \right\} \\
 & + \left\{ \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right. \\
 & \left. - \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \right\} \\
 & \left\{ \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \right. \\
 & \left. - \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 & + \left\{ \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right. \\
 & \left. - \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\}.
 \end{aligned}$$

Since each term is positive we have that

$$B \geq \frac{1}{k^2} \frac{R_1^k}{R_2^k} \left(\varrho_2^k - \frac{1}{\varrho_2^k} \right). \quad (5.20)$$

On the other hand we have:

$$\begin{aligned} |A - e^{ik\vartheta} B| &= \left| \sqrt{N} \{ (\varrho_2^k \lambda_{k+1}^{11} - \lambda_{k+1}^{21}) \langle \overline{\mathbf{l}_2}, \boldsymbol{\xi}_{k+1} \rangle + (\lambda_{k+1}^{22} - \varrho_2^k \lambda_{k+1}^{12}) \langle \overline{\mathbf{l}_1}, \boldsymbol{\xi}_{k+1} \rangle \} \right. \\ &\quad \left. - \{ e^{ik\vartheta} (\lambda_{k+1}^{11} \lambda_{k+1}^{22} - \lambda_{k+1}^{12} \lambda_{k+1}^{21}) \} \right| \end{aligned} \quad (5.21)$$

$$\begin{aligned} &= \frac{N^2}{16\pi^2} \left| \varrho_2^k \sum_{m=0}^{\infty} \left\{ \frac{R_1^{k+mN}}{k+mN} + \frac{R_1^{N-k+mN}}{N-k+mN} \right\} \right. \\ &\quad \times \sum_{m=0}^{\infty} \left\{ \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\ &\quad - \sum_{m=0}^{\infty} \left\{ \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ &\quad \times \sum_{m=0}^{\infty} \left\{ \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\ &\quad + \sum_{m=0}^{\infty} \left\{ \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ &\quad \times \sum_{m=0}^{\infty} \left\{ R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\ &\quad - \varrho_2^k \sum_{m=0}^{\infty} \left\{ \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ &\quad \times \sum_{m=0}^{\infty} \left\{ R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\ &\quad - e^{ik\vartheta} \sum_{m=0}^{\infty} \left\{ \frac{R_1^{k+mN}}{k+mN} + \frac{R_1^{N-k+mN}}{N-k+mN} \right\} \\ &\quad \times \sum_{m=0}^{\infty} \left\{ \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ &\quad + e^{ik\vartheta} \sum_{m=0}^{\infty} \left\{ \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\ &\quad \times \sum_{m=0}^{\infty} \left\{ \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \left| \end{aligned} \quad (5.22)$$

$$\begin{aligned} &= \frac{N^2}{16\pi^2} \left| \varrho_2^k \left\{ \frac{R_1^k}{k} + \frac{R_1^{N-k}}{N-k} + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \right\} \right. \\ &\quad \times \left\{ \frac{1}{R_2^k} \frac{e^{ik\vartheta}}{k} + \frac{1}{R_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{N-k} + \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\ &\quad - \varrho_2^k \left\{ R_1^k \frac{e^{ik\vartheta}}{k} + R_1^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} + \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \end{aligned}$$

$$\begin{aligned}
 & \times \left\{ \frac{1}{R_2^k} \frac{1}{k} + \frac{1}{R_2^{N-k}} \frac{1}{N-k} + \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{1}{N-k+mN} \right\} \\
 & + \left\{ \left(\frac{R_1}{\varrho_2} \right)^k \frac{1}{k} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \frac{1}{N-k} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 & \times \left\{ \frac{1}{R_2^k} \frac{e^{ik\vartheta}}{k} + \frac{1}{R_2^{N-k}} \frac{e^{ik\vartheta}}{N-k} + \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{e^{ik\vartheta}}{k+mN} + \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \\
 & - \left\{ \left(\frac{R_1}{\varrho_2} \right)^k \frac{1}{k} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \frac{1}{N-k} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 & \times \left\{ \frac{1}{R_2^k} \frac{e^{ik\vartheta}}{k} + \frac{1}{R_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{N-k} + \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 & + \left\{ R_1^k \frac{e^{ik\vartheta}}{k} + R_1^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} + \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 & \times \left\{ \left(\frac{\varrho_2}{R_2} \right)^k \frac{1}{k} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{1}{N-k} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 & - \left\{ R_1^k \frac{e^{ik\vartheta}}{k} + R_1^{N-k} \frac{e^{ik\vartheta}}{N-k} + \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{ik\vartheta}}{k+mN} + \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \\
 & \times \left\{ \left(\frac{\varrho_2}{R_2} \right)^k \frac{1}{k} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{1}{N-k} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \Big|_{\infty} \\
 & = \frac{N^2}{16\pi^2} \left\{ \varrho_2^k \left[\frac{R_1^k e^{ik\vartheta}}{R_2^k k^2} + \frac{R_1^k e^{-i(N-k)\vartheta}}{R_2^{N-k} k(N-k)} + \frac{R_1^k}{k} \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \frac{R_1^k}{k} \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right] \right. \\
 & + \varrho_2^k \left[\frac{R_1^{N-k} e^{ik\vartheta}}{R_2^k k(N-k)} + \frac{R_1^{N-k} e^{-i(N-k)\vartheta}}{R_2^{N-k} (N-k)^2} + \frac{R_1^{N-k}}{N-k} \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \\
 & \left. \left. + \frac{R_1^{N-k}}{N-k} \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right] \right\} \\
 & + \varrho_2^k \left\{ \frac{1}{R_2^k} \frac{e^{ik\vartheta}}{k} \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} + \frac{1}{R_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \\
 & \left. + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 & + \varrho_2^k \left\{ \frac{1}{R_2^k} \frac{e^{ik\vartheta}}{k} \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} + \frac{1}{R_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \right. \\
 & \left. + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 & - \varrho_2^k \left\{ \frac{R_1^k e^{ik\vartheta}}{R_2^k k^2} + \frac{R_1^k e^{ik\vartheta}}{R_2^{N-k} k(N-k)} + R_1^k \frac{e^{ik\vartheta}}{k} \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{1}{k+mN} + R_1^k \frac{e^{ik\vartheta}}{k} \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{1}{N-k+mN} \right\}
 \end{aligned}$$

$$\begin{aligned}
& -\varrho_2^k \left\{ \frac{R_1^{N-k} e^{-i(N-k)\vartheta}}{R_2^k k(N-k)} + \frac{R_1^{N-k} e^{-i(N-k)\vartheta}}{R_2^{N-k} (N-k)^2} + R_1^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{1}{k+mN} \right. \\
& \left. + R_1^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{1}{N-k+mN} \right\} \\
& -\varrho_2^k \left\{ \frac{1}{R_2^k} \frac{1}{k} \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \frac{1}{R_2^{N-k}} \frac{1}{N-k} \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \\
& \left. + \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{1}{k+mN} + \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{1}{N-k+mN} \right\} \\
& -\varrho_2^k \left\{ \frac{1}{R_2^k} \frac{1}{k} \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} + \frac{1}{R_2^{N-k}} \frac{1}{N-k} \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right. \\
& \left. + \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \sum_{m=1}^{\infty} \frac{1}{R_2^{k+mN}} \frac{1}{k+mN} \right. \\
& \left. + \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \sum_{m=1}^{\infty} \frac{1}{R_2^{N-k+mN}} \frac{1}{N-k+mN} \right\} \\
& + \left\{ \frac{R_1^k}{\varrho_2^k R_2^{N-k}} \frac{e^{ik\vartheta}}{k(N-k)} + \left(\frac{R_1}{\varrho_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \right. \\
& \left. + \left(\frac{R_1}{\varrho_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \\
& + \left\{ \left(\frac{R_1}{\varrho_2 R_2} \right)^{N-k} \frac{e^{ik\vartheta}}{(N-k)^2} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \right. \\
& \left. + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \\
& + \left\{ \frac{1}{R_2^{N-k}} \frac{e^{ik\vartheta}}{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \right. \\
& + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \\
& + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \left. \right\} \\
& + \left\{ \frac{1}{R_2^{N-k}} \frac{e^{ik\vartheta}}{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right. \\
& + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \\
& + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \left. \right\}
\end{aligned}$$

$$\begin{aligned}
 & - \left\{ \frac{R_1^k}{\varrho_2^k R_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{k(N-k)} + \left(\frac{R_1}{\varrho_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \\
 & \left. + \left(\frac{R_1}{\varrho_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 & - \left\{ \left(\frac{R_1}{\varrho_2 R_2} \right)^{N-k} \frac{e^{-i(N-k)\vartheta}}{(N-k)^2} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \\
 & \left. + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 & - \left\{ \frac{1}{R_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \right. \\
 & + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \\
 & \left. + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 & - \left\{ \frac{1}{R_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right. \\
 & + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \\
 & \left. + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 & + \left\{ \frac{R_1^{N-k} \varrho_2^k}{R_2^k} \frac{e^{-i(N-k)\vartheta}}{k(N-k)} + \left(\frac{R_1 \varrho_2}{R_2} \right)^{N-k} \frac{e^{-i(N-k)\vartheta}}{(N-k)^2} + R_1^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \right. \\
 & \left. + R_1^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 & + \left\{ \left(\frac{\varrho_2}{R_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \\
 & \left. + \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \right. \\
 & \left. + \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 & + \left\{ \left(\frac{\varrho_2}{R_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right. \\
 & + \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \\
 & \left. + \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\}
 \end{aligned}$$

$$\begin{aligned}
& - \left\{ \frac{R_1^{N-k} \varrho_2^k}{R_2^k} \frac{e^{ik\vartheta}}{k(N-k)} + \left(\frac{R_1 \varrho_2}{R_2} \right)^{N-k} \frac{e^{ik\vartheta}}{(N-k)^2} + R_1^{N-k} \frac{e^{ik\vartheta}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \right. \\
& \left. + R_1^{N-k} \frac{e^{ik\vartheta}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\}
\end{aligned}$$

$$\begin{aligned}
& - \left\{ \left(\frac{\varrho_2}{R_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{ik\vartheta}}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \right. \\
& \left. + \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + \sum_{m=1}^{\infty} R_1^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
& - \left\{ \left(\frac{\varrho_2}{R_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{ik\vartheta}}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right. \\
& + \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \\
& \left. + \sum_{m=1}^{\infty} R_1^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\}
\end{aligned}$$

$$\begin{aligned}
 &\leq \frac{2N^2}{16\pi^2} \left\{ \varrho_2^k \left\{ \frac{R_1^k}{R_2^{N-k}} + R_1^k \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} + R_1^k \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} + \frac{R_1^{N-k}}{R_2^k} + \left(\frac{R_1}{R_2} \right)^{N-k} \right. \right. \\
 &\quad + R_1^{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} + R_1^{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} + \frac{1}{R_2^k} \sum_{m=1}^{\infty} R_1^{k+mN} + \frac{1}{R_2^{N-k}} \sum_{m=1}^{\infty} R_1^{k+mN} \\
 &\quad + \sum_{m=1}^{\infty} R_1^{k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} + \sum_{m=1}^{\infty} R_1^{k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} + \frac{1}{R_2^k} \sum_{m=1}^{\infty} R_1^{N-k+mN} \\
 &\quad + \frac{1}{R_2^{N-k}} \sum_{m=1}^{\infty} R_1^{N-k+mN} + \sum_{m=1}^{\infty} R_1^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} + \sum_{m=1}^{\infty} R_1^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \left. \right\} \\
 &\quad + \frac{R_1^k}{\varrho_2^k R_2^{N-k}} + \left(\frac{R_1}{\varrho_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \\
 &\quad + \left(\frac{R_1}{\varrho_2 R_2} \right)^{N-k} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \\
 &\quad + \frac{1}{R_2^{N-k}} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \\
 &\quad + \frac{1}{R_2^{N-k}} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \\
 &\quad + \frac{R_1^{N-k} \varrho_2^k}{R_2^k} + \left(\frac{R_1 \varrho_2}{R_2} \right)^{N-k} + R_1^{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} + R_1^{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \\
 &\quad + \left(\frac{\varrho_2}{R_2} \right)^k \sum_{m=1}^{\infty} R_1^{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \sum_{m=1}^{\infty} R_1^{k+mN} + \sum_{m=1}^{\infty} R_1^{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \\
 &\quad + \sum_{m=1}^{\infty} R_1^{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} + \left(\frac{\varrho_2}{R_2} \right)^k \sum_{m=1}^{\infty} R_1^{N-k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \sum_{m=1}^{\infty} R_1^{N-k+mN} \\
 &\quad + \sum_{m=1}^{\infty} R_1^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} + \sum_{m=1}^{\infty} R_1^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \left. \right\} \\
 &\leq \frac{2N^2}{16\pi^2} \left\{ \varrho_2^k \left\{ \frac{R_1^k}{R_2^{N-k}} + 2R_1^k \left(\frac{1}{R_2} \right)^{k+N} + 2R_1^k \left(\frac{1}{R_2} \right)^{2N-k} + \frac{R_1^{N-k}}{R_2^k} + \left(\frac{R_1}{R_2} \right)^{N-k} + 2R_1^{N-k} \left(\frac{1}{R_2} \right)^{k+N} \right. \right. \\
 &\quad + 2R_1^{N-k} \left(\frac{1}{R_2} \right)^{2N-k} + 2\frac{1}{R_2^k} R_1^{k+N} + 2\frac{1}{R_2^{N-k}} R_1^{k+N} + 4\left(\frac{R_1}{R_2} \right)^{k+N} + 4R_1^{k+N} \left(\frac{1}{R_2} \right)^{2N-k} \\
 &\quad + 2\frac{1}{R_2^k} R_1^{2N-k} + 2\frac{1}{R_2^{N-k}} R_1^{2N-k} + 4R_1^{2N-k} \left(\frac{1}{R_2} \right)^{k+N} + 4\left(\frac{R_1}{R_2} \right)^{2N-k} \left. \right\} + \\
 &\quad + \frac{R_1^k}{\varrho_2^k R_2^{N-k}} + 2\left(\frac{R_1}{\varrho_2} \right)^k \left(\frac{1}{R_2} \right)^{k+N} + 2\left(\frac{R_1}{\varrho_2} \right)^k \left(\frac{1}{R_2} \right)^{2N-k} + \left(\frac{R_1}{\varrho_2 R_2} \right)^{N-k} \\
 &\quad + 2\left(\frac{R_1}{\varrho_2} \right)^{N-k} \left(\frac{1}{R_2} \right)^{k+N} + 2\left(\frac{R_1}{\varrho_2} \right)^{N-k} \left(\frac{1}{R_2} \right)^{2N-k} + 2\left(\frac{R_1}{\varrho_2} \right)^{k+N} \left(\frac{1}{R_2} \right)^{N-k} + 4\left(\frac{R_1}{\varrho_2} \right)^{k+N} \left(\frac{1}{R_2} \right)^{k+N} \\
 &\quad + 4\left(\frac{R_1}{\varrho_2} \right)^{k+N} \left(\frac{1}{R_2} \right)^{2N-k} + 2\left(\frac{R_1}{\varrho_2} \right)^{2N-k} \left(\frac{1}{R_2} \right)^{N-k} + 4\left(\frac{R_1}{\varrho_2} \right)^{2N-k} \left(\frac{1}{R_2} \right)^{k+N} + 4\left(\frac{R_1}{\varrho_2} \right)^{2N-k} \left(\frac{1}{R_2} \right)^{2N-k}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{R_1^{N-k} \varrho_2^k}{R_2^k} + \left(\frac{R_1 \varrho_2}{R_2} \right)^{N-k} + 2R_1^{N-k} \left(\frac{\varrho_2}{R_2} \right)^{k+N} + 2R_1^{N-k} \left(\frac{\varrho_2}{R_2} \right)^{2N-k} + 2R_1^{k+N} \left(\frac{\varrho_2}{R_2} \right)^k \\
& + 2R_1^{k+N} \left(\frac{\varrho_2}{R_2} \right)^{N-k} + 4R_1^{k+N} \left(\frac{\varrho_2}{R_2} \right)^{k+N} + 4R_1^{k+N} \left(\frac{\varrho_2}{R_2} \right)^{2N-k} + 2R_1^{2N-k} \left(\frac{\varrho_2}{R_2} \right)^k + 2R_1^{2N-k} \left(\frac{\varrho_2}{R_2} \right)^{N-k} \\
& + 4R_1^{2N-k} \left(\frac{\varrho_2}{R_2} \right)^{k+N} + 4R_1^{2N-k} \left(\frac{\varrho_2}{R_2} \right)^{2N-k},
\end{aligned}$$

for sufficiently large N . Therefore

$$\begin{aligned}
|A - e^{ik\vartheta} B| & \leq 2N^2 \left\{ \varrho_2^k \left\{ 5R_1^k \left(\frac{1}{R_2} \right)^{N-k} + 10R_1^{N-k} \left(\frac{1}{R_2} \right)^k + 22R_1^{N-k} \left(\frac{1}{R_2} \right)^k \right\} \right. \\
& \quad \left. + 28 \left(\frac{R_1}{R_2} \right)^k \left(\frac{1}{R_2} \right)^{N-k} + 27R_1^k \left(\frac{\varrho_2}{R_2} \right)^{N-k} + 5R_1^{N-k} \left(\frac{\varrho_2}{R_2} \right)^k \right\} \\
& = 2N^2 \left\{ \varrho_2^k \left\{ 5R_1^k \left(\frac{1}{R_2} \right)^{N-k} + 32R_1^{N-k} \left(\frac{1}{R_2} \right)^k \right\} \right. \\
& \quad \left. + 28 \left(\frac{R_1}{R_2} \right)^k \left(\frac{1}{R_2} \right)^{N-k} + 27R_1^k \left(\frac{\varrho_2}{R_2} \right)^{N-k} + 5R_1^{N-k} \left(\frac{\varrho_2}{R_2} \right)^k \right\}.
\end{aligned}$$

Since

$$|B| \geq \frac{1}{k^2} \frac{R_1^k}{R_2^k} \left(\varrho_2^k - \frac{1}{\varrho_2^k} \right) = \frac{1}{k^2} \frac{R_1^k}{R_2^k} \beta \varrho_2^k$$

where $1 > \beta > 1 - \frac{1}{\varrho_2}$, we have

$$\begin{aligned}
\frac{|A - e^{ik\vartheta} B|}{|B|} & \leq \frac{2N^2 k^2}{\beta} \left\{ \frac{5\varrho_2^k R_1^k}{R_2^{N-k}} + \frac{32\varrho_2^k R_1^{N-k}}{R_2^{N-k}} + \frac{28R_1^k}{R_2^k R_2^{N-k}} + \frac{27\varrho_2^{N-k} R_1^k}{R_2^{N-k}} + \frac{5\varrho_2^k R_1^{N-k}}{R_2^k} \right\} \\
& = \frac{2N^2 k^2}{\beta} \left\{ \frac{5}{R_2^{N-2k}} + 32R_1^{N-2k} + \frac{28}{\varrho_2^k R_2^{N-k}} + 27 \left(\frac{\varrho_2}{R_2} \right)^{N-2k} + 5R_1^{N-2k} \right\}
\end{aligned}$$

If we take $\omega = \max \left\{ R_1, \frac{1}{R_2}, \frac{\varrho_2}{R_2} \right\} = \max \left\{ R_1, \frac{\varrho_2}{R_2} \right\} < 1$, we have

$$\begin{aligned}
\frac{|A - e^{ik\vartheta} B|}{|B|} & \leq \frac{2N^2 k^2}{\beta} \left\{ 5\omega^{N-2k} + 32\omega^{N-2k} + 28\omega^{N-2k} + 27\omega^{N-2k} + 5\omega^{N-2k} \right\} \\
& = \frac{2N^2 k^2}{\beta} 97\omega^{N-2k} = \frac{194N^2 k^2}{\beta} \omega^{N-2k},
\end{aligned}$$

where $1 > \beta > 1 - \frac{1}{\varrho_2}$.

Case III, $-N/2 \leq k < 0$: From 4.2 we have

$$\begin{aligned}
u_N & = \sum_{j=1}^N \frac{\lambda_j^{11}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_j \rangle} - \sum_{j=1}^N \frac{\lambda_j^{21}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_j \rangle} \\
& \quad - \sum_{j=1}^N \frac{\lambda_j^{12}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_j \rangle} + \sum_{j=1}^N \frac{\lambda_j^{22}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_j \rangle},
\end{aligned}$$

$$\langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle = \begin{cases} \sqrt{N} & \text{if } j = k+1 \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle = \begin{cases} \sqrt{N} \frac{1}{\varrho_2^{|k|}} & \text{if } j = k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$u_N = \frac{\sqrt{N}}{\lambda_{k+1}^{11} \lambda_{k+1}^{22} - \lambda_{k+1}^{12} \lambda_{k+1}^{21}} \left\{ \left(\frac{1}{\varrho_2^{|k|}} \lambda_{k+1}^{11} - \lambda_{k+1}^{21} \right) \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_{k+1} \rangle} + \left(\lambda_{k+1}^{22} - \frac{1}{\varrho_2^{|k|}} \lambda_{k+1}^{12} \right) \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_{k+1} \rangle} \right\}$$

Therefore, since $u(\cdot; \bar{h}) = \overline{u(\cdot; h)}$ and $u_N(\cdot; \bar{h}) = \overline{u_N(\cdot; h)}$, we have

$$\left\| u_N(\cdot; e^{-i|k|\varphi}) - u(\cdot; e^{-i|k|\varphi}) \right\|_{\infty} = \max_{\vartheta \in [0, 2\pi]} \left| \frac{A}{B} - e^{-i|k|\vartheta} \right| = \max_{\vartheta \in [0, 2\pi]} \frac{|A - e^{-i|k|\vartheta} B|}{|B|}$$

where

$$A = \sqrt{N} \left\{ \left(\frac{1}{\varrho_2^{|k|}} \lambda_{k+1}^{11} - \lambda_{k+1}^{21} \right) \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_{k+1} \rangle} + \left(\lambda_{k+1}^{22} - \frac{1}{\varrho_2^{|k|}} \lambda_{k+1}^{12} \right) \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_{k+1} \rangle} \right\}$$

and

$$B = \lambda_{k+1}^{11} \lambda_{k+1}^{22} - \lambda_{k+1}^{12} \lambda_{k+1}^{21}$$

Following a similar proof as in case II with the following differences in $|A - e^{i|k|\vartheta} B|$:

- The coefficient $\varrho_2^{|k|}$ in equation (5.21) is replaced by $\frac{1}{\varrho_2^{|k|}}$,
- The angles in (5.22) have opposite signs.

We obtain

$$\frac{|A - e^{-i|k|\vartheta} B|}{|B|} \leq \frac{194N^2 k^2}{\beta} \omega^{N-2|k|}$$

where $1 > \beta > 1 - \frac{1}{\varrho_2}$.

Next, we bound the term $u_N(\cdot; r^k e^{ik\varphi}) - u(\cdot; r^k e^{ik\varphi})$ on the circle $r = \varrho_2$.

Case I. $k = 0$. In this case, $f_1 = f_2 = 1$, $\mathbf{f}_1 = \mathbf{f}_2 = (1, 1, \dots, 1)$, $u \equiv 1$ and

$$\langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle = \begin{cases} \sqrt{N} & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle = \begin{cases} \sqrt{N} & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus from expression (4.2) we have

$$u_N(\cdot; 1) = \frac{\sqrt{N}}{\lambda_1^{11} \lambda_1^{22} - \lambda_1^{12} \lambda_1^{21}} \left\{ (\lambda_1^{11} - \lambda_1^{21}) \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_1 \rangle} + (\lambda_1^{22} - \lambda_1^{12}) \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_1 \rangle} \right\}$$

where the eigenvalues take the form:

$$\lambda_1^{11} = \frac{N}{2\pi} \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN}, \quad \lambda_1^{12} = \frac{N}{2\pi} \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} - \log R_2 \right\}$$

$$\lambda_1^{21} = \frac{N}{2\pi} \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} - \log \varrho_2 \right\}, \quad \lambda_1^{22} = \frac{N}{2\pi} \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} - \log R_2 \right\}$$

and

$$\overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_1 \rangle} = \frac{\sqrt{N}}{2\pi} \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} \cos(mN\vartheta) - \log \varrho_2 \right\},$$

$$\overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_1 \rangle} = \frac{\sqrt{N}}{2\pi} \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right\}$$

Therefore

$$\|u_N(\cdot; 1) - u(\cdot; 1)\|_\infty = \max_{\vartheta \in [0, 2\pi]} |u_N(\cos \vartheta, \sin \vartheta; 1) - 1| = \max_{\vartheta \in [0, 2\pi]} \frac{|A - B|}{|B|},$$

where

$$A = \frac{N}{2\pi} \left\{ (\lambda_1^{11} - \lambda_1^{21}) \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right\} \right. \\ \left. + (\lambda_1^{22} - \lambda_1^{12}) \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} \cos(mN\vartheta) - \log \varrho_2 \right\} \right\}$$

and $B = \lambda_1^{11} \lambda_1^{22} - \lambda_1^{12} \lambda_1^{21}$. We thus have

$$|B| = \frac{1}{4\pi^2} \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \left\{ N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \right\} \\ + \left\{ N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{R_2} \right)^{mN} \right\} \left\{ N \log \varrho_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right\} \geq N^2 \log R_2 \log \varrho_2$$

and

$$|A - B| = \left| \frac{N}{2\pi} \left\{ (\lambda_1^{11} - \lambda_1^{21}) \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right\} \right. \right. \\ \left. \left. + (\lambda_1^{22} - \lambda_1^{12}) \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} \cos(mN\vartheta) - \log \varrho_2 \right\} \right\} - (\lambda_1^{11} \lambda_1^{22} - \lambda_1^{12} \lambda_1^{21}) \right| \\ = \left| \frac{N}{2\pi} \left\{ \left(\frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} + \frac{1}{2\pi} \left[N \log \varrho_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right] \right) \right. \right. \\ \left. \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right) \right. \\ \left. + \left(\frac{1}{2\pi} \left[\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} - N \log R_2 \right] - \frac{1}{2\pi} \left[\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{R_2} \right)^{mN} - N \log R_2 \right] \right) \right. \\ \left. \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} \cos(mN\vartheta) - \log \varrho_2 \right) \right\} \\ + \frac{1}{4\pi^2} \left\{ \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \left[N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \right] \right. \\ \left. + \left[N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{R_2} \right)^{mN} \right] \left[N \log \varrho_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right] \right\} \right|$$

$$\begin{aligned}
 &= \frac{N}{4\pi^2} \left| \left\{ \left(\sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} + N \log \varrho_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right) \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right) \right. \right. \\
 &\quad + \left(\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{R_2^{mN}} \right) \\
 &\quad \left. \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} \cos(mN\vartheta) - \log \varrho_2 \right) \right\} \\
 &\quad + \frac{1}{4\pi^2} \left\{ \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \left[N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \right] \right. \\
 &\quad \left. + \left[N \log R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{R_2} \right)^{mN} \right] \left[N \log \varrho_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right] \right\} \Big|
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{4\pi^2} \left| \left\{ N \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} \cos(mN\vartheta) - N \log R_2 \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \right. \right. \\
 &\quad + N^2 \log \varrho_2 \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} \cos(mN\vartheta) - N^2 \log \varrho_2 \log R_2 \\
 &\quad - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \cos(mN\vartheta) \\
 &\quad + N \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \log R_2 \\
 &\quad + N \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} \cos(mN\vartheta) \\
 &\quad - N \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \log \varrho_2 \\
 &\quad - N \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{R_2} \right)^{mN} \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \cos(mN\vartheta) - N \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{R_2} \right)^{mN} \log \varrho_2 \\
 &\quad + N \log R_2 \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} - \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_2}{R_2} \right)^{mN} \\
 &\quad + N^2 \log \varrho_2 \log R_2 - N \log R_2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} - N \log \varrho_2 \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{R_2^{mN}} \\
 &\quad \left. + \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{R_2^{mN}} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right\} \Big|
 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{m=1}^{\infty} R_1^{mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2}\right)^{mN} + N \log R_2 \sum_{m=1}^{\infty} R_1^{mN} + N^2 \log \varrho_2 \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2}\right)^{mN} \\
&\quad + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2}\right)^{mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2}\right)^{mN} + N \log R_2 \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2}\right)^{mN} \\
&\quad + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2}\right)^{mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2}\right)^{mN} + N \log \varrho_2 \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2}\right)^{mN} \\
&\quad + \sum_{m=1}^{\infty} \frac{1}{R_2^{mN}} \sum_{m=1}^{\infty} R_1^{mN} + N \log \varrho_2 \sum_{m=1}^{\infty} \frac{1}{R_2^{mN}} \\
&\quad + N \log R_2 \sum_{m=1}^{\infty} R_1^{mN} + \sum_{m=1}^{\infty} R_1^{mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2}\right)^{mN} + N \log R_2 \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2}\right)^{mN} \\
&\quad + N \log \varrho_2 \sum_{m=1}^{\infty} \frac{1}{R_2^{mN}} + \sum_{m=1}^{\infty} \frac{1}{R_2^{mN}} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2}\right)^{mN} \\
&\leq 2R_1^N 2 \left(\frac{\varrho_2}{R_2}\right)^N + N \log R_2 2R_1^N + N^2 \log \varrho_2 2 \left(\frac{\varrho_2}{R_2}\right)^N + \\
&\quad 2 \left(\frac{R_1}{\varrho_2}\right)^N \frac{2}{R_2^N} 2 \left(\frac{\varrho_2}{R_2}\right)^N \\
&\quad + N \log R_2 2 \left(\frac{R_1}{\varrho_2}\right)^N + 4 \left(\frac{\varrho_2}{R_2}\right)^N R_1^N + 2N \log \varrho_2 \left(\frac{\varrho_2}{R_2}\right)^N + \frac{2}{R_2^N} 2R_1^N + N \log \varrho_2 \frac{2}{R_2^N} \\
&\quad + 2N \log R_2 R_1^N + 4R_1^N \left(\frac{\varrho_2}{R_2}\right)^N + 2N \log R_2 \left(\frac{R_1}{\varrho_2}\right)^N + 2N \log R_2 \frac{1}{R_2^N} + 4 \frac{1}{R_2^N} \left(\frac{R_1}{\varrho_2}\right)^N.
\end{aligned}$$

We take $\omega = \max \left\{ R_1, \frac{\varrho_2}{R_2} \right\} < 1$ and therefore:

$$\begin{aligned}
|A - B| &\leq 4\omega^{2N} + 2N \log R_2 \omega^N + 2N^2 \log R_2 \omega^N + 4\omega^{2N} + 2N \log R_2 \omega^N + 4\omega^{2N} + 2N \log \varrho_2 \omega^N \\
&\quad + 4\omega^{2N} + 2N \log \varrho_2 \omega^N + 2N \log R_2 \omega^N + 4\omega^{2N} + 2N \log R_2 \omega^N + 4N \log R_2 \omega^N + 4\omega^{2N} \\
&\leq 4\omega^N + 2N \log R_2 \omega^N + 2N^2 \log R_2 \omega^N + 4\omega^N + 2N \log R_2 \omega^N + 4\omega^N + 2N \log R_2 \omega^N + 4\omega^N \\
&\quad + 2N \log R_2 \omega^N + 4\omega^N + 2N \log R_2 \omega^N + 4\omega^N + 4N \log \varrho_2 \omega^N \\
&\leq 24\omega^N + 18N^2 \log R_2 \omega^N.
\end{aligned}$$

Thus if $\log R_2 > 1$ we have,

$$\begin{aligned}
\frac{|A - B|}{|B|} &\leq \frac{24\omega^N + 18N^2 \log R_2 \omega^N}{N^2 \log R_2 \log \varrho_2} \leq \frac{24N \log R_2 \omega^N + 18N^2 \log R_2 \omega^N}{N^2 \log R_2 \log \varrho_2} \leq \frac{42N^2 \log R_2 \omega^N}{N^2 \log R_2 \log \varrho_2} \\
&= \frac{42\omega^N}{\log \varrho_2}.
\end{aligned}$$

If $\log R_2 < 1$ we have:

$$\begin{aligned}
\frac{|A - B|}{|B|} &\leq \frac{24\omega^N + 18N^2 \log R_2 \omega^N}{N^2 \log R_2 \log \varrho_2} \leq \frac{24N\omega^N + 18N^2\omega^N}{N^2 \log R_2 \log \varrho_2} = \frac{42\omega^N}{\log R_2 \log \varrho_2} \\
&= \frac{42\omega^N}{\log \varrho_2 \log R_2}.
\end{aligned}$$

Case II, $0 < k \leq N/2$: From (4.2) we have:

$$\begin{aligned}
 u_N = & \sum_{j=1}^N \frac{\lambda_j^{11}}{\lambda_j^{11}\lambda_j^{22} - \lambda_j^{12}\lambda_j^{21}} \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_j \rangle} - \sum_{j=1}^N \frac{\lambda_j^{21}}{\lambda_j^{11}\lambda_j^{22} - \lambda_j^{12}\lambda_j^{21}} \langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_j \rangle} \\
 & - \sum_{j=1}^N \frac{\lambda_j^{12}}{\lambda_j^{11}\lambda_j^{22} - \lambda_j^{12}\lambda_j^{21}} \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_j \rangle} + \sum_{j=1}^N \frac{\lambda_j^{22}}{\lambda_j^{11}\lambda_j^{22} - \lambda_j^{12}\lambda_j^{21}} \langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_j \rangle}
 \end{aligned}$$

where

$$\langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle = \begin{cases} \sqrt{N} & \text{if } j = k+1 \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle = \begin{cases} \sqrt{N}\varrho_2^k & \text{if } j = k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$u_N = \frac{\sqrt{N}}{\lambda_{k+1}^{11}\lambda_{k+1}^{22} - \lambda_{k+1}^{12}\lambda_{k+1}^{21}} \{ (\varrho_2^k \lambda_{k+1}^{11} - \lambda_{k+1}^{21}) \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_{k+1} \rangle} + (\lambda_{k+1}^{22} - \varrho_2^k \lambda_{k+1}^{12}) \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_{k+1} \rangle} \}.$$

We can therefore write

$$|u_N - \varrho_2^k e^{ik\theta}| = \left| \frac{A}{B} - \varrho_2^k e^{ik\theta} \right| = \frac{|A - \varrho_2^k e^{ik\theta} B|}{|B|} \quad (5.23)$$

where

$$A = \sqrt{N} \{ (\varrho_2^k \lambda_{k+1}^{11} - \lambda_{k+1}^{21}) \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_{k+1} \rangle} + (\lambda_{k+1}^{22} - \varrho_2^k \lambda_{k+1}^{12}) \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_{k+1} \rangle} \}$$

and

$$B = \lambda_{k+1}^{11}\lambda_{k+1}^{22} - \lambda_{k+1}^{12}\lambda_{k+1}^{21}.$$

Since $\varrho_1 = 1$, we have

$$\begin{aligned}
 \lambda_{k+1}^{11} &= \frac{N}{4\pi} \sum_{m=0}^{\infty} \left\{ R_1^{k+mN} \frac{1}{k+mN} + R_1^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 \lambda_{k+1}^{12} &= \frac{N}{4\pi} \sum_{m=0}^{\infty} \left\{ \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 \lambda_{k+1}^{21} &= \frac{N}{4\pi} \sum_{m=0}^{\infty} \left\{ \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 \lambda_{k+1}^{22} &= \frac{N}{4\pi} \sum_{m=0}^{\infty} \left\{ \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\}.
 \end{aligned}$$

From equation (5.20) we have that

$$B > \frac{1}{k^2} \frac{R_1^k}{R_2^k} \left(\varrho_2^k - \frac{1}{\varrho_2^k} \right).$$

On the other hand we have

$$\begin{aligned}
& |A - \varrho_2^k e^{ik\vartheta} B| \\
&= \left| \sqrt{N} \{ (\varrho_2^k \lambda_{k+1}^{11} - \lambda_{k+1}^{21}) \overline{(\mathbf{l}_2, \boldsymbol{\xi}_{k+1})} + (\lambda_{k+1}^{22} - \varrho_2^k \lambda_{k+1}^{12}) \overline{(\mathbf{l}_1, \boldsymbol{\xi}_{k+1})} \} - \varrho_2^k e^{ik\vartheta} (\lambda_{k+1}^{11} \lambda_{k+1}^{22} - \lambda_{k+1}^{12} \lambda_{k+1}^{21}) \right| \\
&= \frac{N^2}{16\pi^2} \left| \varrho_2^k \sum_{m=0}^{\infty} \left\{ \frac{R_1^{k+mN}}{k+mN} + \frac{R_1^{N-k+mN}}{N-k+mN} \right\} \right. \\
&\quad \times \sum_{m=0}^{\infty} \left\{ \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
&\quad - \sum_{m=0}^{\infty} \left\{ \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
&\quad \times \sum_{m=0}^{\infty} \left\{ \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
&\quad + \sum_{m=0}^{\infty} \left\{ \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
&\quad \times \sum_{m=0}^{\infty} \left\{ \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
&\quad - \varrho_2^k \sum_{m=0}^{\infty} \left\{ \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
&\quad \times \sum_{m=0}^{\infty} \left\{ \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
&\quad - \varrho_2^k e^{ik\vartheta} \sum_{m=0}^{\infty} \left\{ \frac{R_1^{k+mN}}{k+mN} + \frac{R_1^{N-k+mN}}{N-k+mN} \right\} \\
&\quad \times \sum_{m=0}^{\infty} \left\{ \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
&\quad + \varrho_2^k e^{ik\vartheta} \sum_{m=0}^{\infty} \left\{ \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
&\quad \times \sum_{m=0}^{\infty} \left\{ \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \left. \right|
\end{aligned}$$

$$\begin{aligned}
 &= \frac{N^2}{16\pi^2} \left| \varrho_2^k \left\{ \frac{R_1^k}{k} + \frac{R_1^{N-k}}{N-k} + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \right\} \right. \\
 &\quad \times \left\{ \left(\frac{\varrho_2}{R_2} \right)^k \frac{e^{ik\vartheta}}{k} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} \right. \\
 &\quad \left. + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 &\quad - \varrho_2^k \left\{ \frac{R_1^k}{k} + \frac{R_1^{N-k}}{N-k} + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \right\} \\
 &\quad \times \left\{ \left(\frac{\varrho_2}{R_2} \right)^k \frac{e^{ik\vartheta}}{k} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{e^{ik\vartheta}}{N-k} \right. \\
 &\quad \left. + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \\
 &\quad + \left\{ \left(\frac{\varrho_2}{R_2} \right)^k \frac{1}{k} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{1}{N-k} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 &\quad \times \left\{ \left(\frac{R_1}{\varrho_2} \right)^k \frac{e^{ik\vartheta}}{k} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 &\quad - \left\{ \left(\frac{\varrho_2}{R_2} \right)^k \frac{e^{ik\vartheta}}{k} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \\
 &\quad \left. + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 &\quad \times \left\{ \left(\frac{R_1}{\varrho_2} \right)^k \frac{1}{k} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \frac{1}{N-k} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 &\quad + \varrho_2^k e^{ik\vartheta} \left\{ \frac{1}{kR_2^k} + \frac{1}{(N-k)R_2^{N-k}} + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 &\quad \times \left\{ \left(\frac{R_1}{\varrho_2} \right)^k \frac{1}{k} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \frac{1}{N-k} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 &\quad - \varrho_2^k \left\{ \frac{1}{kR_2^k} + \frac{1}{(N-k)R_2^{N-k}} + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 &\quad \times \left\{ \left(\frac{R_1}{\varrho_2} \right)^k \frac{e^{ik\vartheta}}{k} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \\
 &\quad \left. + \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{i(N-k+mN)\vartheta}}{N-k+mN} \right\} \left| \right.
 \end{aligned}$$

$$\begin{aligned}
&= \frac{N^2}{16\pi^2} \left| \varrho_2^k \left\{ \frac{R_1^k \varrho_2^{N-k}}{R_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{k(N-k)} + \frac{R_1^k}{k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \right. \\
&\quad \left. \left. + \frac{R_1^k}{k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \right. \\
&\quad \left. + \varrho_2^k \left\{ \frac{R_1^{N-k} \varrho_2^{N-k}}{R_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{(N-k)^2} + \frac{R_1^{N-k}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \right. \\
&\quad \left. \left. + \frac{R_1^{N-k}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \right. \\
&\quad \left. + \varrho_2^k \left\{ \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \right. \\
&\quad \left. \left. + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \right. \\
&\quad \left. + \varrho_2^k \left\{ \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \right. \\
&\quad \left. \left. + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \right. \\
&\quad \left. - \varrho_2^k \left\{ \frac{R_1^k \varrho_2^{N-k}}{R_2^{N-k}} \frac{e^{ik\vartheta}}{k(N-k)} + \frac{R_1^k}{k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \right. \right. \\
&\quad \left. \left. + \frac{R_1^k}{k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \right. \\
&\quad \left. - \varrho_2^k \left\{ \frac{R_1^{N-k} \varrho_2^{N-k}}{R_2^{N-k}} \frac{e^{ik\vartheta}}{(N-k)^2} + \frac{R_1^{N-k}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \right. \right. \\
&\quad \left. \left. + \frac{R_1^{N-k}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \right. \\
&\quad \left. - \varrho_2^k \left\{ \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{e^{ik\vartheta}}{N-k} \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \right. \right. \\
&\quad \left. \left. + \sum_{m=1}^{\infty} \frac{R_1^{k+mN}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \right. \\
&\quad \left. - \varrho_2^k \left\{ \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{e^{ik\vartheta}}{N-k} \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \right. \right. \\
&\quad \left. \left. + \sum_{m=1}^{\infty} \frac{R_1^{N-k+mN}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \right. \\
&\quad \left. + \left\{ \frac{\varrho_2^k R_1^k}{R_2^k \varrho_2^k} \frac{e^{ik\vartheta}}{k^2} + \frac{\varrho_2^k R_1^{N-k}}{R_2^k \varrho_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{k(N-k)} + \left(\frac{\varrho_2}{R_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \right. \\
&\quad \left. \left. + \left(\frac{\varrho_2}{R_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \right. \\
&\quad \left. + \left\{ \frac{\varrho_2^{N-k} R_1^k}{R_2^{N-k} \varrho_2^k} \frac{e^{ik\vartheta}}{k(N-k)} + \frac{\varrho_2^{N-k} R_1^{N-k}}{R_2^{N-k} \varrho_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{(N-k)^2} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \right. \\
&\quad \left. \left. + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \right.
\end{aligned}$$

$$\begin{aligned}
 & + \left\{ \frac{R_1^k}{\varrho_2^k} \frac{e^{ik\vartheta}}{k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} + \frac{R_1^{N-k}}{\varrho_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \right. \\
 & + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \\
 & + \left. \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 & + \left\{ \frac{R_1^k}{\varrho_2^k} \frac{e^{ik\vartheta}}{k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} + \frac{R_1^{N-k}}{\varrho_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right. \\
 & + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \\
 & + \left. \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
 & - \left\{ \frac{\varrho_2^k R_1^k}{R_2^k \varrho_2^k} \frac{e^{ik\vartheta}}{k^2} + \frac{\varrho_2^k R_1^{N-k}}{R_2^k \varrho_2^{N-k}} \frac{e^{ik\vartheta}}{k(N-k)} + \left(\frac{\varrho_2}{R_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \right. \\
 & + \left. \left(\frac{\varrho_2}{R_2} \right)^k \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \\
 & - \left\{ \frac{\varrho_2^{N-k} R_1^k}{R_2^{N-k} \varrho_2^k} \frac{e^{-i(N-k)\vartheta}}{k(N-k)} + \frac{\varrho_2^{N-k} R_1^{N-k}}{R_2^{N-k} \varrho_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{(N-k)^2} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{-i(N-k)\vartheta}}{k+mN} \right. \\
 & + \left. \left(\frac{\varrho_2}{R_2} \right)^{N-k} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k)\vartheta}}{N-k+mN} \right\} \\
 & - \left\{ \frac{R_1^k}{\varrho_2^k} \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} + \frac{R_1^{N-k}}{\varrho_2^{N-k}} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \\
 & + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \\
 & + \left. \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 & - \left\{ \frac{R_1^k}{\varrho_2^k} \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} + \frac{R_1^{N-k}}{\varrho_2^{N-k}} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right. \\
 & + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{1}{k+mN} \\
 & + \left. \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \right\} \\
 & + \varrho_2^k \left\{ \frac{R_1^k}{R_2^k \varrho_2^k} \frac{e^{ik\vartheta}}{k^2} + \frac{R_1^{N-k}}{R_2^k \varrho_2^{N-k}} \frac{e^{ik\vartheta}}{k(N-k)} + \frac{1}{k R_2^k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \right. \\
 & + \left. \frac{1}{k R_2^k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \varrho_2^k \left\{ \frac{R_1^k}{R_2^{N-k} \varrho_2^k} \frac{e^{ik\vartheta}}{k(N-k)} + \frac{R_1^{N-k}}{R_2^{N-k} \varrho_2^{N-k}} \frac{e^{ik\vartheta}}{(N-k)^2} + \frac{1}{(N-k)R_2^{N-k}} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \right. \\
& + \left. \frac{1}{(N-k)R_2^{N-k}} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \\
& + \varrho_2^k \left\{ \frac{R_1^k}{\varrho_2^k} \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} + \frac{R_1^{N-k}}{\varrho_2^{N-k}} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \right. \\
& + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \\
& + \left. \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \\
& + \varrho_2^k \left\{ \frac{R_1^k}{\varrho_2^k} \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} + \frac{R_1^{N-k}}{\varrho_2^{N-k}} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right. \\
& + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} \\
& + \left. \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} \right\} \\
& - \varrho_2^k \left\{ \frac{R_1^k}{R_2^k \varrho_2^k} \frac{e^{ik\vartheta}}{k^2} + \frac{R_1^{N-k}}{R_2^k \varrho_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{k(N-k)} + \frac{1}{kR_2^k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \\
& + \left. \frac{1}{kR_2^k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
& - \varrho_2^k \left\{ \frac{R_1^k}{R_2^{N-k} \varrho_2^k} \frac{e^{ik\vartheta}}{k(N-k)} + \frac{R_1^{N-k}}{R_2^{N-k} \varrho_2^{N-k}} \frac{e^{-i(N-k)\vartheta}}{(N-k)^2} + \frac{1}{(N-k)R_2^{N-k}} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \right. \\
& + \left. \frac{1}{(N-k)R_2^{N-k}} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \\
& - \varrho_2^k \left\{ \frac{R_1^k}{\varrho_2^k} \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{ik\vartheta}}{k+mN} + \frac{R_1^{N-k}}{\varrho_2^{N-k}} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{e^{-i(N-k)\vartheta}}{k+mN} \right. \\
& + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \\
& + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \frac{1}{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \left. \right\} \\
& - \varrho_2^k \left\{ \frac{R_1^k}{\varrho_2^k} \frac{1}{k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{ik\vartheta}}{N-k+mN} + \frac{R_1^{N-k}}{\varrho_2^{N-k}} \frac{1}{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{e^{-i(N-k)\vartheta}}{N-k+mN} \right. \\
& + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \frac{e^{i(k+mN)\vartheta}}{k+mN} \\
& + \left. \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \frac{1}{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \frac{e^{-i(N-k+mN)\vartheta}}{N-k+mN} \right\} \Big|
\end{aligned}$$

$$\begin{aligned}
 &\leq \frac{2N^2}{16\pi^2} \varrho_2^k \left\{ \frac{R_1^k \varrho_2^{N-k}}{R_2^{N-k}} + R_1^k \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} + R_1^k \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} + \frac{R_1^{N-k} \varrho_2^{N-k}}{R_2^{N-k}} \right. \\
 &\quad + R_1^{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} + R_1^{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \\
 &\quad + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \sum_{m=1}^{\infty} R_1^{k+mN} + \sum_{m=1}^{\infty} R_1^{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \\
 &\quad + \sum_{m=1}^{\infty} R_1^{k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \sum_{m=1}^{\infty} R_1^{N-k+mN} + \sum_{m=1}^{\infty} R_1^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \\
 &\quad \left. + \sum_{m=1}^{\infty} R_1^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \right\} \\
 &\quad + \frac{\varrho_2^k R_1^{N-k}}{R_2^k \varrho_2^{N-k}} + \left(\frac{\varrho_2}{R_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} + \frac{\varrho_2^{N-k} R_1^k}{R_2^{N-k} \varrho_2^k} \\
 &\quad + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} + \left(\frac{\varrho_2}{R_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} + \left(\frac{R_1}{\varrho_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \\
 &\quad + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \\
 &\quad + \left(\frac{R_1}{\varrho_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \\
 &\quad + \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \\
 &\quad + \varrho_2^k \left\{ \left(\frac{R_1}{\varrho_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} \right. \\
 &\quad + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} + \left(\frac{R_1}{\varrho_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \\
 &\quad + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{N-k+mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \\
 &\quad + \frac{R_1^{N-k}}{R_2^k \varrho_2^{N-k}} + \left(\frac{1}{R_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} + \left(\frac{1}{R_2} \right)^k \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \\
 &\quad \left. + \frac{R_1^k}{R_2^{N-k} \varrho_2^k} + \frac{R_1^{N-k}}{R_2^{N-k} \varrho_2^{N-k}} + \left(\frac{1}{R_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{k+mN} + \left(\frac{1}{R_2} \right)^{N-k} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{N-k+mN} \right\}
 \end{aligned}$$

$$\begin{aligned}
&\leq 2N^2 \varrho_2^k \left\{ \frac{R_1^k \varrho_2^{N-k}}{R_2^{N-k}} + 2R_1^k \left(\frac{\varrho_2}{R_2} \right)^{k+N} + 2R_1^k \left(\frac{\varrho_2}{R_2} \right)^{2N-k} + R_1^{N-k} \left(\frac{\varrho_2}{R_2} \right)^{N-k} + 2R_1^{N-k} \left(\frac{\varrho_2}{R_2} \right)^{k+N} \right. \\
&\quad + 2R_1^{N-k} \left(\frac{\varrho_2}{R_2} \right)^{2N-k} + 2R_1^{k+N} \left(\frac{\varrho_2}{R_2} \right)^{N-k} + 4R_1^{k+N} \left(\frac{\varrho_2}{R_2} \right)^{k+N} + 4R_1^{k+N} \left(\frac{\varrho_2}{R_2} \right)^{2N-k} + 2R_1^{2N-k} \left(\frac{\varrho_2}{R_2} \right)^{N-k} \\
&\quad \left. + 4R_1^{2N-k} \left(\frac{\varrho_2}{R_2} \right)^{k+N} + 4R_1^{2N-k} \left(\frac{\varrho_2}{R_2} \right)^{2N-k} \right\} \\
&\quad + \frac{\varrho_2^k R_1^{N-k}}{R_2^k \varrho_2^{N-k}} + 2 \left(\frac{\varrho_2}{R_2} \right)^k \left(\frac{R_1}{\varrho_2} \right)^{k+N} + 2 \left(\frac{\varrho_2}{R_2} \right)^k \left(\frac{R_1}{\varrho_2} \right)^{2N-k} \\
&\quad + \frac{\varrho_2^{N-k} R_1^k}{R_2^{N-k} \varrho_2^k} + 2 \left(\frac{\varrho_2}{R_2} \right)^{N-k} \left(\frac{R_1}{\varrho_2} \right)^{k+N} + 2 \left(\frac{\varrho_2}{R_2} \right)^{N-k} \left(\frac{R_1}{\varrho_2} \right)^{2N-k} + 2 \left(\frac{\varrho_2}{R_2} \right)^{k+N} \left(\frac{R_1}{\varrho_2} \right)^k \\
&\quad + 2 \left(\frac{\varrho_2}{R_2} \right)^{k+N} \left(\frac{R_1}{\varrho_2} \right)^{N-k} + 4 \left(\frac{\varrho_2}{R_2} \right)^{k+N} \left(\frac{R_1}{\varrho_2} \right)^{k+N} + 4 \left(\frac{\varrho_2}{R_2} \right)^{k+N} \left(\frac{R_1}{\varrho_2} \right)^{2N-k} + 2 \left(\frac{\varrho_2}{R_2} \right)^{2N-k} \left(\frac{R_1}{\varrho_2} \right)^k \\
&\quad + 2 \left(\frac{\varrho_2}{R_2} \right)^{2N-k} \left(\frac{R_1}{\varrho_2} \right)^{N-k} + 4 \left(\frac{\varrho_2}{R_2} \right)^{2N-k} \left(\frac{R_1}{\varrho_2} \right)^{k+N} + 4 \left(\frac{\varrho_2}{R_2} \right)^{2N-k} \left(\frac{R_1}{\varrho_2} \right)^{2N-k} \\
&\quad + \varrho_2^k \left\{ 2 \left(\frac{R_1}{\varrho_2} \right)^k \left(\frac{1}{R_2} \right)^{k+N} + 2 \left(\frac{R_1}{\varrho_2} \right)^{N-k} \left(\frac{1}{R_2} \right)^{k+N} + 4 \left(\frac{R_1}{\varrho_2} \right)^{k+N} \left(\frac{1}{R_2} \right)^{k+N} + 4 \left(\frac{R_1}{\varrho_2} \right)^{2N-k} \left(\frac{1}{R_2} \right)^{k+N} \right. \\
&\quad + 2 \left(\frac{R_1}{\varrho_2} \right)^k \left(\frac{1}{R_2} \right)^{2N-k} + 2 \left(\frac{R_1}{\varrho_2} \right)^{N-k} \left(\frac{1}{R_2} \right)^{2N-k} + 4 \left(\frac{R_1}{\varrho_2} \right)^{k+N} \left(\frac{1}{R_2} \right)^{2N-k} + 4 \left(\frac{R_1}{\varrho_2} \right)^{2N-k} \left(\frac{1}{R_2} \right)^{2N-k} \\
&\quad + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \left(\frac{1}{R_2} \right)^k + 2 \left(\frac{R_1}{\varrho_2} \right)^{k+N} \left(\frac{1}{R_2} \right)^k + 2 \left(\frac{R_1}{\varrho_2} \right)^{2N-k} \left(\frac{1}{R_2} \right)^k + \left(\frac{R_1}{\varrho_2} \right)^k \left(\frac{1}{R_2} \right)^{N-k} \\
&\quad \left. + \left(\frac{R_1}{\varrho_2} \right)^{N-k} \left(\frac{1}{R_2} \right)^{N-k} + 2 \left(\frac{R_1}{\varrho_2} \right)^{k+N} \left(\frac{1}{R_2} \right)^{N-k} + 2 \left(\frac{R_1}{\varrho_2} \right)^{2N-k} \left(\frac{1}{R_2} \right)^{N-k} \right\}
\end{aligned}$$

for sufficiently large N . Therefore

$$\begin{aligned}
|A - \varrho_2^k e^{ik\vartheta} B| &\leq 2N^2 \left\{ \varrho_2^k 30R_1^k \left(\frac{\varrho_2}{R_2} \right)^{N-k} + 29 \left(\frac{\varrho_2}{R_2} \right)^k \left(\frac{R_1}{\varrho_2} \right)^{N-k} + 5 \left(\frac{\varrho_2}{R_2} \right)^{N-k} \left(\frac{R_1}{\varrho_2} \right)^k \right. \\
&\quad \left. + \varrho_2^k \left\{ 30 \left(\frac{R_1}{\varrho_2} \right)^k \left(\frac{1}{R_2} \right)^{N-k} + 5 \left(\frac{R_1}{\varrho_2} \right)^{N-k} \left(\frac{1}{R_2} \right)^k \right\} \right\}.
\end{aligned}$$

Since

$$|B| \geq \frac{1}{k^2} \frac{R_1^k}{R_2^k} \left(\varrho_2^k - \frac{1}{\varrho_2^k} \right) = \frac{1}{k^2} \frac{R_1^k}{R_2^k} \beta \varrho_2^k,$$

where $1 > \beta > 1 - \frac{1}{\varrho_2}$, we have

$$\begin{aligned}
\frac{|A - e^{ik\vartheta} B|}{|B|} &\leq \frac{2N^2 k^2}{\beta} \left\{ \frac{30 \varrho_2^k R_1^k \varrho_2^{N-k}}{R_2^{N-k}} + \frac{29 \varrho_2^k R_1^{N-k}}{R_2^k \varrho_2^{N-k}} + \frac{5 \varrho_2^{N-k} R_1^k}{\varrho_2^k R_2^{N-k}} + \frac{30 \varrho_2^k R_1^k}{\varrho_2^k R_2^{N-k}} + \frac{5 \varrho_2^k R_1^{N-k}}{\varrho_2^{N-k} R_2^k} \right\} \\
&= \frac{2N^2 k^2}{\beta} \left\{ 30R_2^k \left(\frac{\varrho_2}{R_2} \right)^{N-k} + \frac{29}{R_1^k} \left(\frac{R_1}{\varrho_2} \right)^{N-k} + \frac{5}{\varrho_2^k} \left(\frac{\varrho_2}{R_2} \right)^{N-2k} + \frac{30}{\varrho_2^k R_2^{N-2k}} + \frac{5}{R_1^k} \left(\frac{R_1}{\varrho_2} \right)^{N-k} \right\} \\
&\leq \frac{2N^2 k^2}{\beta} 99R_2^k \left(\frac{\varrho_2}{R_2} \right)^{N-k} = \frac{198N^2 k^2}{\beta} R_2^k \left(\frac{\varrho_2}{R_2} \right)^{N-k}
\end{aligned}$$

where $1 > \beta > 1 - \frac{1}{\varrho_2}$.

Case III, $-N/2 \leq k < 0$:

The proof is analogous to the proof of case II in the case when the maximum is obtained on the external circle ϱ_2 . As in (5.23) we now have:

$$\begin{aligned} \left| u_N - \frac{1}{\varrho_2^{|k|}} e^{-i|k|\vartheta} \right| &= \left| u_N \left(\frac{1}{\varrho_2^{|k|}} e^{-i|k|\vartheta} \right) - \frac{1}{\varrho_2^{|k|}} e^{-i|k|\vartheta} \right| = \left| \overline{u_N} \left(\frac{1}{\varrho_2^{|k|}} e^{i|k|\vartheta} \right) - \frac{1}{\varrho_2^{|k|}} e^{-i|k|\vartheta} \right| \\ &= \left| \frac{A}{B} - \frac{1}{\varrho_2^{|k|}} e^{-i|k|\vartheta} \right| = \frac{\left| A - \frac{1}{\varrho_2^{|k|}} e^{-i|k|\vartheta} B \right|}{|B|}, \end{aligned}$$

where

$$A = \sqrt{N} \left\{ \left(\frac{1}{\varrho_2^{|k|}} \lambda_{k+1}^{11} - \lambda_{k+1}^{21} \right) \langle \overline{\mathbf{l}_2}, \boldsymbol{\xi}_{k+1} \rangle + \left(\lambda_{k+1}^{22} - \frac{1}{\varrho_2^{|k|}} \lambda_{k+1}^{12} \right) \langle \overline{\mathbf{l}_1}, \boldsymbol{\xi}_{k+1} \rangle \right\}$$

and

$$B = \lambda_{k+1}^{11} \lambda_{k+1}^{22} - \lambda_{k+1}^{12} \lambda_{k+1}^{21}.$$

Thus

$$\frac{\left| A - \frac{1}{\varrho_2^{|k|}} e^{-i|k|\vartheta} B \right|}{|B|} \leq \frac{\left| A - \frac{1}{\varrho_2^{|k|}} e^{-i|k|\vartheta} B \right|}{\frac{1}{|k|^2} \frac{R_1^{|k|}}{R_2^{|k|}} \left(\varrho_2^{|k|} - \frac{1}{\varrho_2^k} \right)} = \frac{\left| A - \frac{1}{\varrho_2^{|k|}} e^{-i|k|\vartheta} B \right|}{\frac{1}{|k|^2} \frac{R_1^{|k|}}{R_2^{|k|}} \beta \varrho_2^{|k|}}$$

where $1 > \beta > 1 - \frac{1}{\varrho_2}$. Therefore

$$\begin{aligned} & \frac{\left| A - \frac{1}{\varrho_2^{|k|}} e^{-i|k|\vartheta} B \right|}{|B|} \\ & \leq \frac{2N^2 |k|^2}{\beta} \left\{ \frac{30 R_1^{|k|} \varrho_2^{N-|k|}}{R_2^{N-|k|} \varrho_2^k} + \frac{29 \varrho_2^k R_1^{N-|k|}}{R_2^{|k|} \varrho_2^{N-|k|}} + \frac{5 \varrho_2^{N-|k|} R_1^{|k|}}{\varrho_2^{|k|} R_2^{N-|k|}} + \frac{30 R_1^{|k|}}{\varrho_2^{|k|} \varrho_2^{|k|} R_2^{N-|k|}} + \frac{5 R_1^{N-|k|}}{\varrho_2^{|k|} \varrho_2^{N-|k|} R_2^{|k|}} \right\} \\ & = \frac{2N^2 k^2}{\beta} \left\{ 30 \frac{\varrho_2^{N-|k|} R_2^{|k|}}{\varrho_2^{2|k|} R_2^{N-|k|}} + 29 \frac{R_1^{N-|k|}}{R_1^{|k|} \varrho_2^{N-|k|}} + 5 \frac{R_2^{|k|} \varrho_2^{N-|k|}}{R_2^{N-|k|} \varrho_2^{2|k|}} + 30 \frac{R_2^{|k|}}{\varrho_2^{2|k|} \varrho_2^{|k|} R_2^{N-|k|}} + 5 \frac{R_1^{N-|k|}}{\varrho_2^{2|k|} R_1^{|k|} \varrho_2^{N-|k|}} \right\} \\ & = \frac{2N^2 k^2}{\beta} \left\{ \frac{30}{\varrho_2^{|k|}} \left(\frac{\varrho_2}{R_2} \right)^{N-2|k|} + \frac{29}{\varrho_2^{|k|}} \left(\frac{R_1}{\varrho_2} \right)^{N-2|k|} + \frac{5}{\varrho_2^{|k|}} \left(\frac{\varrho_2}{R_2} \right)^{N-2|k|} + \frac{30}{\varrho_2^{3|k|}} \frac{1}{R_2^{N-2|k|}} + \frac{5}{\varrho_2^{3|k|}} \left(\frac{R_1}{R_2} \right)^{N-2|k|} \right\} \\ & \leq \frac{2N^2 k^2}{\beta} \left\{ 30 \left(\frac{\varrho_2}{R_2} \right)^{N-2|k|} + 29 \left(\frac{R_1}{\varrho_2} \right)^{N-2|k|} + 5 \left(\frac{\varrho_2}{R_2} \right)^{N-2|k|} + 30 \left(\frac{1}{R_2} \right)^{N-2|k|} + 5 \left(\frac{R_1}{R_2} \right)^{N-2|k|} \right\} \\ & = \frac{2N^2 k^2}{\beta} \left\{ 35 \left(\frac{\varrho_2}{R_2} \right)^{N-2|k|} + 29 \left(\frac{R_1}{\varrho_2} \right)^{N-2|k|} + 30 \left(\frac{1}{R_2} \right)^{N-2|k|} + 5 \left(\frac{R_1}{R_2} \right)^{N-2|k|} \right\}. \end{aligned}$$

By taking $\omega = \max \left\{ \frac{\varrho_2}{R_2}, \frac{1}{R_2}, \frac{R_1}{\varrho_2}, \frac{R_1}{R_2} \right\} = \max \left\{ \frac{\varrho_2}{R_2}, \frac{R_1}{\varrho_2} \right\} < 1$, we have that

$$\frac{\left| A - \frac{1}{\varrho_2^{|k|}} e^{-i|k|\vartheta} B \right|}{|B|} \leq \frac{198N^2 k^2}{\beta} \omega^{N-2|k|}$$

where $1 > \beta > 1 - \frac{1}{\varrho_2}$.

We now find a bound for the term $u_N(\cdot; \log r) - u(\cdot; \log r)$:

Lemma 5.5. *We have the following estimates:*

$$|u_N(\cdot, \log r) - u(\cdot, \log r)|_\infty \leq \frac{c}{N} \omega^N \quad (5.24)$$

where $\omega = \max\left\{R_1, \frac{1}{R_2}\right\} < 1$ and c is a constant independent of N .

Proof. Case I: When the maximum is obtained on the inner circle $r = \varrho_1 = 1$, from expression (4.2) we have:

$$\begin{aligned} u_N &= \sum_{j=1}^N \frac{\lambda_j^{11}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_j \rangle} - \sum_{j=1}^N \frac{\lambda_j^{21}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_j \rangle} \\ &\quad - \sum_{j=1}^N \frac{\lambda_j^{12}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_j \rangle} + \sum_{j=1}^N \frac{\lambda_j^{22}}{\lambda_j^{11} \lambda_j^{22} - \lambda_j^{12} \lambda_j^{21}} \langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_j \rangle} \end{aligned}$$

where

$$\langle \mathbf{f}_1, \boldsymbol{\xi}_j \rangle = 0 \quad \text{and} \quad \langle \mathbf{f}_2, \boldsymbol{\xi}_j \rangle = \begin{cases} \sqrt{N} \log \varrho_2 & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$u_N = \frac{\sqrt{N}}{\lambda_1^{11} \lambda_1^{22} - \lambda_1^{12} \lambda_1^{21}} \{ \log \varrho_2 \lambda_1^{11} \overline{\langle \mathbf{l}_2, \boldsymbol{\xi}_1 \rangle} - \log \varrho_2 \lambda_1^{12} \overline{\langle \mathbf{l}_1, \boldsymbol{\xi}_1 \rangle} \},$$

and therefore

$$|u_N - \log \varrho_1| = \left| \frac{A}{B} - 0 \right| = \left| \frac{A}{B} \right|$$

where

$$\begin{aligned} |A| &= \frac{N^2}{4\pi^2} \left| \log \varrho_2 \left\{ \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right) \right. \right. \\ &\quad \left. \left. - \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} - \log R_2 \right) \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \cos(mN\vartheta) \right\} \right| \\ &= \frac{N}{4\pi^2} \left| \log \varrho_2 \left\{ \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \right. \right. \\ &\quad \left. \left. - \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \cos(mN\vartheta) \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} + \log R_2 \sum_{m=1}^{\infty} \frac{1}{m} R_1^{mN} \cos(mN\vartheta) \right\} \right| \\ &\leq \frac{N}{4\pi^2} \log \varrho_2 \left\{ \frac{1}{N} \sum_{m=1}^{\infty} R_1^{mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{mN} + \log R_2 \sum_{m=1}^{\infty} R_1^{mN} \right. \\ &\quad \left. + \frac{1}{N} \sum_{m=1}^{\infty} R_1^{mN} \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{mN} + \log R_2 \sum_{m=1}^{\infty} R_1^{mN} \right\} \\ &\leq \log \varrho_2 \left\{ 4R_1^N \frac{1}{R_2^N} + 2N \log R_2 R_1^N + 4R_1^N \frac{1}{R_2^N} + 2N \log R_2 R_1^N \right\} \\ &= \log \varrho_2 \left\{ 8 \left(\frac{R_1}{R_2} \right)^N + 4N \log R_2 R_1^N \right\} \\ &\leq \log \varrho_2 \{ 8NR_1^N + 4N \log R_2 R_1^N \} \end{aligned}$$

for sufficiently large N . In the case $\log R_2 > 1$, we have

$$\frac{|A|}{|B|} \leq \frac{\log \varrho_2 \log R_2 (8NR_1^N + 4NR_1^N)}{N^2 \log \varrho_2 \log R_2} = \frac{12R_1^N}{N},$$

whereas in the case $\log R_2 < 1$, we have

$$\frac{|A|}{|B|} \leq \frac{\log \varrho_2 (8NR_1^N + 4NR_1^N)}{N^2 \log \varrho_2 \log R_2} = \frac{12R_1^N}{N \log R_2}$$

Case II: When the maximum is obtained on the external circle $r = \varrho_2$ we have

$$|u_N - \log \varrho_2| = \left| \frac{A}{B} - \log \varrho_2 \right| = \frac{|A - \log \varrho_2 B|}{|B|},$$

where

$$\begin{aligned}
|A - \log \varrho_2 B| &= \frac{N^2}{4\pi^2} \left| \log \varrho_2 \lambda_1^{11} \langle \overline{\mathbf{l}_2}, \boldsymbol{\xi}_1 \rangle - \log \varrho_2 \lambda_1^{12} \langle \overline{\mathbf{l}_1}, \boldsymbol{\xi}_1 \rangle - \log \varrho_2 (\lambda_1^{11} \lambda_1^{22} - \lambda_1^{12} \lambda_1^{21}) \right| \\
&= \log \varrho_2 \frac{N^2}{4\pi^2} \left| \lambda_1^{11} \langle \overline{\mathbf{l}_2}, \boldsymbol{\xi}_1 \rangle - \lambda_1^{12} \langle \overline{\mathbf{l}_1}, \boldsymbol{\xi}_1 \rangle - \lambda_1^{11} \lambda_1^{22} + \lambda_1^{12} \lambda_1^{21} \right| \\
&= \log \varrho_2 \frac{N^2}{4\pi^2} \left| \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right) \right. \\
&\quad \left. - \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} - \log R_2 \right) \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} \cos(mN\vartheta) - \log \varrho_2 \right) \right. \\
&\quad \left. - \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \right) \right. \\
&\quad \left. + \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} - \log R_2 \right) \left(\sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} - \log \varrho_2 \right) \right| \\
&= \log \varrho_2 \frac{N^2}{4\pi^2} \left| \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} \cos(mN\vartheta) - \log R_2 \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \right. \\
&\quad \left. - \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} \cos(mN\vartheta) + \log \varrho_2 \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} \right. \\
&\quad \left. + \log R_2 \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} \cos(mN\vartheta) - \log \varrho_2 \log R_2 - \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{\varrho_2}{R_2} \right)^{mN} \right. \\
&\quad \left. + \log R_2 \sum_{m=1}^{\infty} \frac{1}{mN} R_1^{mN} + \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} - \log \varrho_2 \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{1}{R_2} \right)^{mN} \right. \\
&\quad \left. - \log R_2 \sum_{m=1}^{\infty} \frac{1}{mN} \left(\frac{R_1}{\varrho_2} \right)^{mN} + \log \varrho_2 \log R_2 \right| \\
&\leq \log \varrho_2 \left\{ \sum_{m=1}^{\infty} R_1^{mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{mN} + N \log R_2 \sum_{m=1}^{\infty} R_1^{mN} + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right. \\
&\quad \left. + N \log \varrho_2 \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{mN} + N \log R_2 \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{mN} + \sum_{m=1}^{\infty} R_1^{mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{mN} \right. \\
&\quad \left. + N \log R_2 \sum_{m=1}^{\infty} R_1^{mN} + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{mN} + N \log \varrho_2 \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{mN} \right. \\
&\quad \left. + N \log R_2 \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right\} \\
&= 2 \log \varrho_2 \left\{ \sum_{m=1}^{\infty} R_1^{mN} \sum_{m=1}^{\infty} \left(\frac{\varrho_2}{R_2} \right)^{mN} + N \log R_2 \sum_{m=1}^{\infty} R_1^{mN} + \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{mN} \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right. \\
&\quad \left. + N \log \varrho_2 \sum_{m=1}^{\infty} \left(\frac{1}{R_2} \right)^{mN} + N \log R_2 \sum_{m=1}^{\infty} \left(\frac{R_1}{\varrho_2} \right)^{mN} \right\} \\
&\leq 2 \log \varrho_2 \left\{ 4R_1^N \left(\frac{\varrho_2}{R_2} \right)^N + 2N \log R_2 R_1^N + 4 \frac{1}{R_2^N} \left(\frac{R_1}{\varrho_2} \right)^N + 2N \log \varrho_2 \frac{1}{R_2^N} + 2N \log R_2 \left(\frac{R_1}{\varrho_2} \right)^N \right\} \\
&\leq 2 \log \varrho_2 \{ 4\omega^{2N} + 2N \log R_2 \omega^N + 4\omega^{2N} + 2N \log R_2 \omega^N + 2N \log R_2 \omega^N \} \\
&\leq \log \varrho_2 \{ 16\omega^N + 12N \log R_2 \omega^N \} \\
&\leq \log \varrho_2 \omega^N N \{ 16 + 12 \log R_2 \}
\end{aligned}$$

for sufficient large N and taking $\omega = \max\{R_1, \varrho_2/R_2\}$. In the case $\log R_2 > 1$, we have

$$\frac{|A - \log \varrho_2 B|}{|B|} \leq \frac{28 \log \varrho_2 \log R_2 N \omega^N}{N^2 \log \varrho_2 \log R_2} = \frac{28 \omega^N}{N}.$$

whereas when $\log R_2 < 1$, we have

$$\frac{|A - \log \varrho_2 B|}{|B|} \leq \frac{28 \log \varrho_2 N \omega^N}{N^2 \log \varrho_2 \log R_2} = \frac{28 \omega^N}{N \log R_2}.$$

We are now in a position to obtain a bound for $u_N(\cdot, f_N) - u(\cdot, f_N)$ by summing the bounds we have obtained so far.

$$\begin{aligned} \|u_N(\cdot, f_N) - u(\cdot, f_N)\|_\infty &\leq \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} |\mu_k| \|u_N(r^k e^{ik\vartheta}) - u(r^k e^{ik\vartheta})\|_\infty \\ &\quad + \sum_{k=-\frac{N}{2}+1, k \neq 0}^{\frac{N}{2}} |\nu_k| \|u_N(r^{-k} e^{ik\vartheta}) - u(r^{-k} e^{ik\vartheta})\|_\infty \\ &\quad + |\nu_0| \|u_N(\cdot, \log r) - u(\cdot, \log r)\|_\infty. \end{aligned}$$

Using Theorem (5.11) and Lemmata (5.19), (5.24) we obtain

$$\begin{aligned} \|u_N(\cdot, f_N) - u(\cdot, f_N)\|_\infty &\leq \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} M \gamma_4^{|k|} c N^4 \omega^{N-2|k|} + \sum_{k=-\frac{N}{2}+1, k \neq 0}^{\frac{N}{2}} M \gamma_4^{|k|} c N^4 \omega^{N-2|k|} + M \frac{c}{N} \omega^N \\ &\leq 2cMN^4 \omega^N \sum_{k=0}^{\frac{N}{2}} \left(\frac{\gamma_4}{\omega^2}\right)^k \leq 4cMN^4 \omega^N \frac{N}{4} \left\{1 + \left(\frac{\gamma_4}{\omega^2}\right)^{\frac{N}{2}}\right\} \\ &= cMN^5 \left(\omega^N + \gamma_4^{\frac{N}{2}}\right) \end{aligned}$$

5.2.3. The desired bound. We take $\gamma = \max\{\gamma_3, \gamma_4\} < 1$. From the bounds on $\|u(\cdot; \mathbf{f} - \mathbf{f}_N)\|$ and $\|u_N(\cdot; \mathbf{f}_N) - u(\cdot; \mathbf{f}_N)\|$ and inequality (5.12) given by (5.18) and (5.2.2), respectively, we get the desired bound

Theorem 5.6.

$$\|u_N(\cdot; f) - u(\cdot; f)\|_\infty \leq CMN^5 \left(\omega^N + \gamma^{\frac{N}{2}}\right). \quad (5.25)$$

6. NUMERICAL RESULTS

We considered the following numerical examples corresponding to the Dirichlet problem (1.1) in the annulus defined by $\varrho_1 = 1, \varrho_2 = 2$:

Example 1. Problem corresponding to the exact solution $u = e^x \cos y$. We varied the angle of α and examined how this affected the accuracy of the solution for various values of N for different $\varepsilon = \varrho_1 - R_1 = R_2 - \varrho_2$ (Figures 1, 2). We also varied the radii R_1 and R_2 of the inner and external circle and examined how this affected the accuracy of the solution for various values of N (Figure 7). In addition, we varied the radius R_2 of the external circle and examined how this affected the accuracy of the solution for various values of N (Figure 10).

Example 2. Problem corresponding to the exact solution $u = \frac{x}{x^2 + y^2}$. We varied the angle α and examined how this affected the accuracy of the solution for various values of N for different $\varepsilon = \varrho_1 - R_1 =$

$R_2 - \varrho_2$ (Figures 3, 4). We also varied the radii R_1 and R_2 of the inner and external circle and examined how this affected the accuracy of the solution for various values of N (Figure 8).

Example 3. Problem corresponding to the exact solution $u = x^2 - y^2$. We varied the angle of α and examined how this affected the accuracy of the solution for various values of N for different $\varepsilon = \varrho_1 - R_1 = R_2 - \varrho_2$ (Figures 5, 6). We also varied the radii R_1 and R_2 of the inner and external circle and examined how this affected the accuracy of the solution for various values of N (Figure 9).

The maximum relative error in these examples was calculated on a uniform grid of m points on the boundary (since all the functions involved are harmonic and the maximum principle applies) defined by

$$\begin{aligned} (\varrho_1 \cos \vartheta_j, \varrho_1 \sin \vartheta_j), \quad \vartheta_j &= \frac{2\pi(j-1)}{m}, \quad j = 1, \dots, m \\ (\varrho_2 \cos \vartheta_j, \varrho_2 \sin \vartheta_j), \quad \vartheta_j &= \frac{2\pi(j-1)}{m}, \quad j = 1, \dots, m \end{aligned}$$

The parameter m is taken to be equal to 500.

The numerical results indicate that for small ε the accuracy of the solution is dependent on the angular parameter α . As ε grows this dependence disappears. Further, the accuracy for a certain range of values of ε reaches an optimal value for $\alpha = 0.25$. This phenomenon was also been observed in [9, 10] and is valid for all the examples considered in this paper (Figures 1-6). Also, we observed that as ε increases for $\alpha = 0$, the accuracy of the method improves (Figures 7-9) according to the theoretical predictions developed in this study. However, for larger values of ε the accuracy deteriorates due to ill-conditioning. This is more evident in Figure 10, when we kept the inner pseudoboundary fixed and equal to 0.5 and increased only the outer pseudoboundary.

From Figures 7, 8 and 9 it is evident that as R_1 decreases, R_2 increases and N increases the MFS solution converges exponentially fast to the exact solution for all the test problems. The same phenomenon can be observed for Figure 10, where R_1 is fixed and R_2 increases. However in all cases, there is a deterioration of accuracy for large values of R_2 and N due to ill-conditioning.

Note. We have also considered the case when the two pseudoboundaries are rotated independently. In particular, we take the coordinates of the points $Q_j = (x_{Q_j}, y_{Q_j})$ on the internal pseudo-boundary to be

$$x_{Q_j} = R_1 \cos \frac{2(j-1+\alpha_1)\pi}{N}, \quad y_{Q_j} = R_1 \sin \frac{2(j-1+\alpha_1)\pi}{N}, \quad j = 1, \dots, N$$

and the coordinates of the points $Q_{N+j} = (x_{Q_{N+j}}, y_{Q_{N+j}})$ on the external pseudo-boundary to be

$$x_{Q_{N+j}} = R_2 \cos \frac{2(j-1+\alpha_2)\pi}{N}, \quad y_{Q_{N+j}} = R_2 \sin \frac{2(j-1+\alpha_2)\pi}{N}, \quad j = 1, \dots, N.$$

The independent rotation produced little difference in the numerical results.

7. CONCLUSIONS

In this study we examine the application of the MFS to harmonic problems in annular domains subject to Dirichlet boundary conditions. The properties of the coefficient matrix are investigated, and an efficient algorithm for the numerical solution of the problem is proposed. It is shown that, for analytic boundary data, the MFS approximation converges exponentially to the exact solution. The results of the current investigation can be also applied to other second order elliptic operators such as the Helmholtz operator for problems in annular domains. Further, the application of the algorithm proposed in this paper to the biharmonic equation in angular domains is being examined. The extension of the ideas developed in

this study to the solution of harmonic and biharmonic problems in shell type axisymmetric domains (see [11]), is currently under investigation.

REFERENCES

- [1] P. J. DAVIS, *Circulant Matrices*, John Wiley & Sons, New York, 1979.
- [2] G. FAIRWEATHER AND A. KARAGEORGHIS, *The method of fundamental solutions for elliptic boundary value problems*, Adv. Comput. Math., 9, 69-95, 1998.
- [3] G. FAIRWEATHER, A. KARAGEORGHIS AND P. A. MARTIN, *The method of fundamental solutions for scattering and radiation problems*, Engng. Analysis with Boundary Elements, 27, 759-769, 2003.
- [4] M. A. GOLBERG AND C. S. CHEN, *Discrete Projection Methods for Integral Equations*, Computational Mechanics Publications, Southampton, 1996.
- [5] I.S. GRADSHTEYN AND I.M. RYZHIK, *Table of Integrals, Series, and Products*, 6th edition, Academic Press, London, 2000.
- [6] M. KATSURADA, *A mathematical study of the charge simulation method II*, J. Fac. Sci., Univ. of Tokyo, Sect. 1A, Math. 36, 135-162, 1989.
- [7] M. KATSURADA AND H.OKAMOTO, *A mathematical study of the charge simulation method I*, J. Fac. Sci., Univ. of Tokyo, Sect. 1A, Math. 35, 507-518, 1988.
- [8] *Numerical Algorithms Group Library Mark 20*, NAG Ltd, Wilkinson House, Jordan Aill Road, Oxford, UK, 2001.
- [9] Y.S. SMYRLIS AND A. KARAGEORGHIS, *Some aspects of the method of fundamental solutions for certain harmonic problems*, J. Sci. Comput., 16, 341-371, 2001.
- [10] Y.S. SMYRLIS AND A. KARAGEORGHIS, *Numerical analysis of the MFS for certain harmonic problems*, Technical Report TR-04-2003, Department of Mathematics and Statistics, University of Cyprus.
- [11] Y.S. SMYRLIS AND A. KARAGEORGHIS, *A matrix decomposition MFS algorithm for axisymmetric potential problems*, Engng. Analysis with Boundary Elements, 28, 463-474, 2004.
- [12] W. A. STRAUSS, *Partial Differential Equations. An Introduction*, John Wiley & Sons, Inc., New York, 1992.

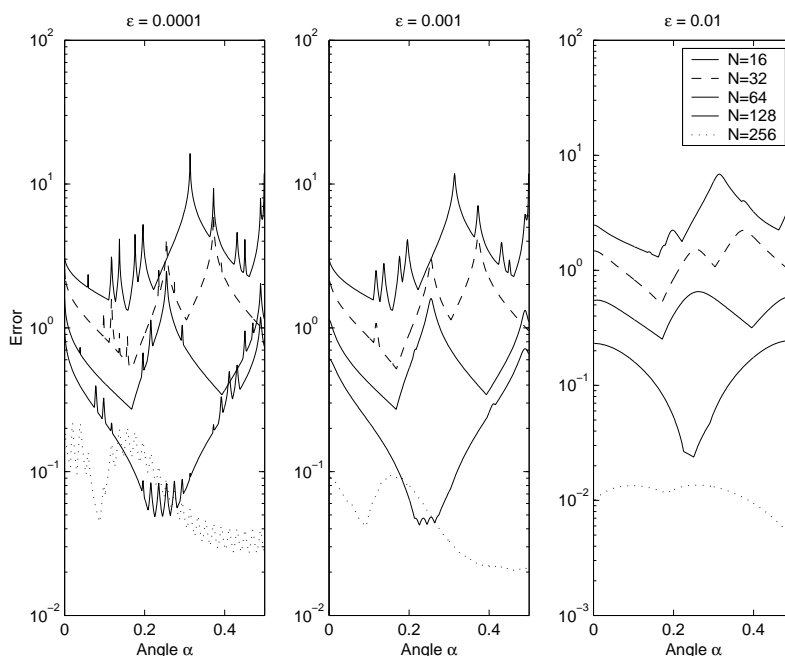


FIGURE 1. Log-plot of error versus angular parameter α for $\epsilon = 10^{-4}, 10^{-3}, 10^{-2}$ in Example 1 for different values of N

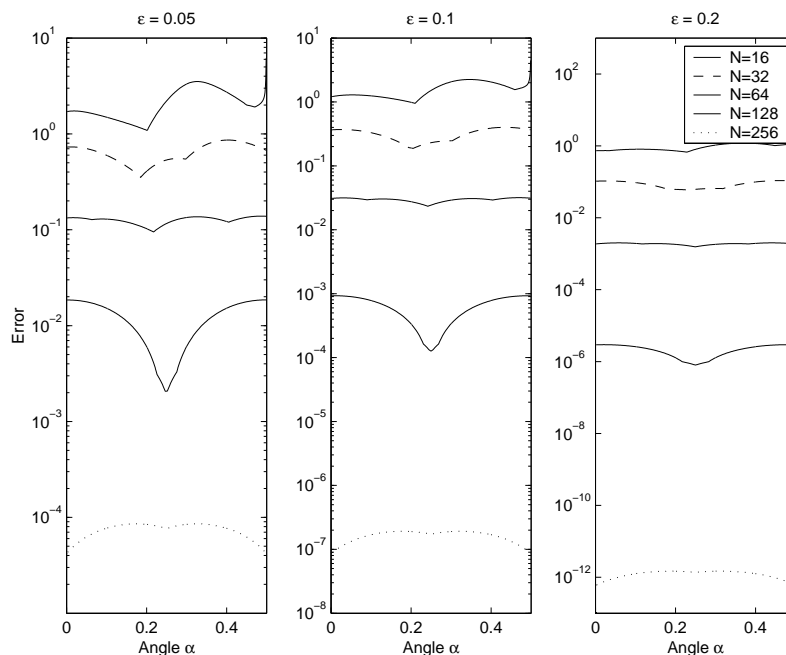


FIGURE 2. Log-plot of error versus angular parameter α for $\epsilon = 0.05, 0.1, 0.2$ in Example 1 for different values of N

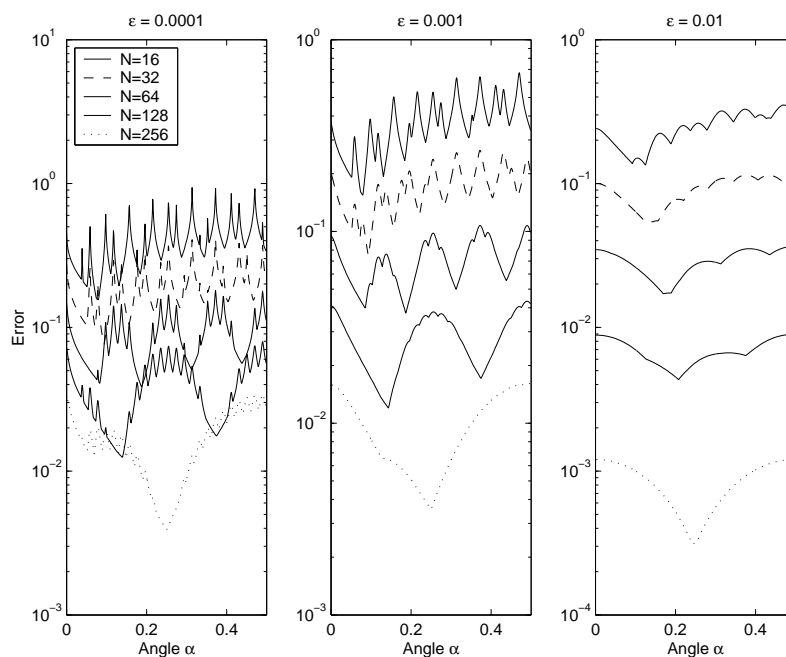


FIGURE 3. Log-plot of error versus angular parameter α for $\epsilon = 10^{-4}, 10^{-3}, 10^{-2}$ in Example 2 for different values of N

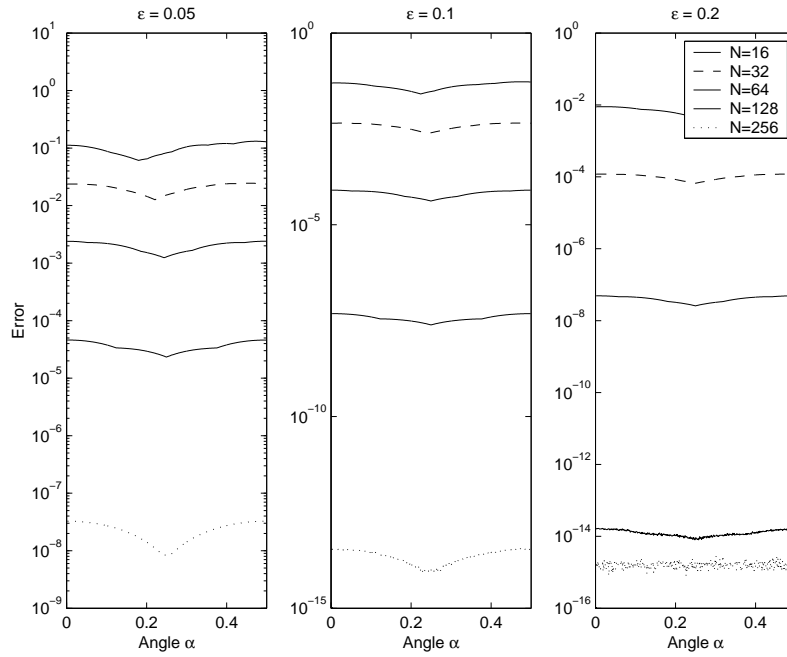


FIGURE 4. Log-plot of error versus angular parameter α for $\epsilon = 0.05, 0.1, 0.2$ in Example 2 for different values of N

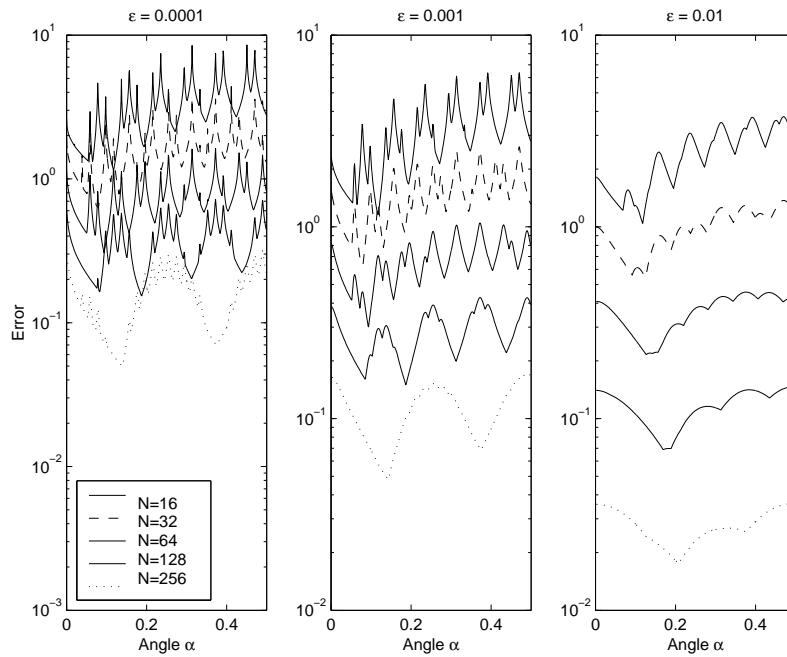


FIGURE 5. Log-plot of error versus angular parameter α for $\epsilon = 10^{-4}, 10^{-3}, 10^{-2}$ in Example 3 for different values of N

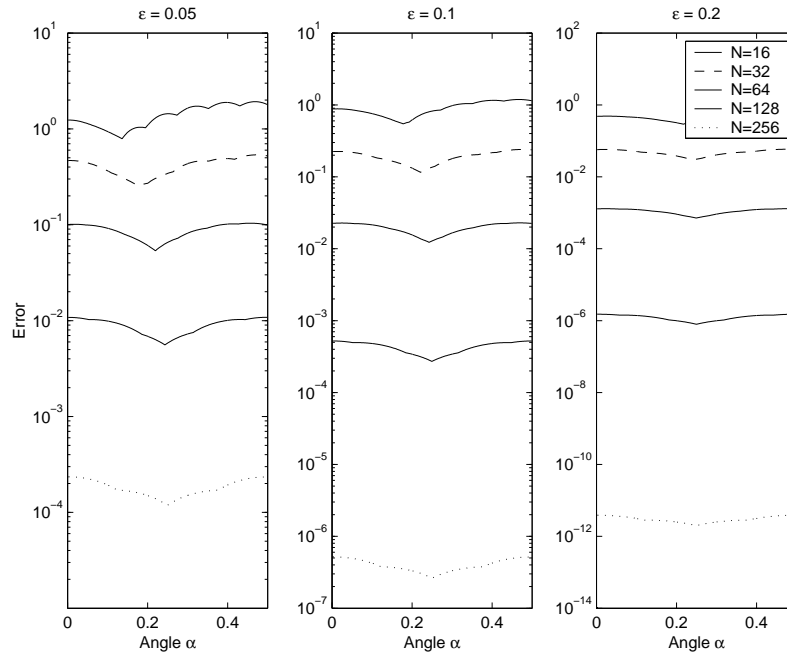


FIGURE 6. Log-plot of error versus angular parameter α for $\epsilon = 0.05, 0.1, 0.2$ in Example 3 for different values of N

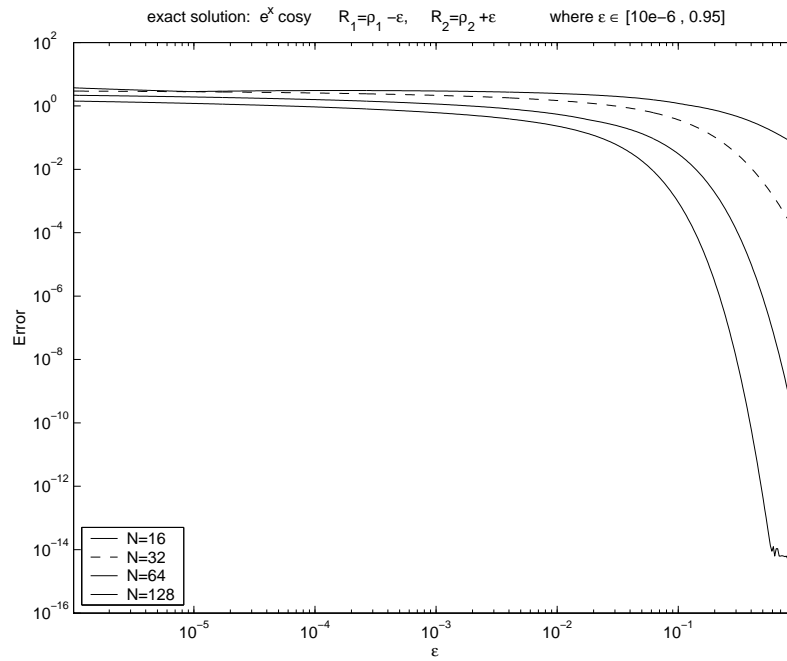


FIGURE 7. Log-log plot of maximum relative error versus ϵ in Example 1

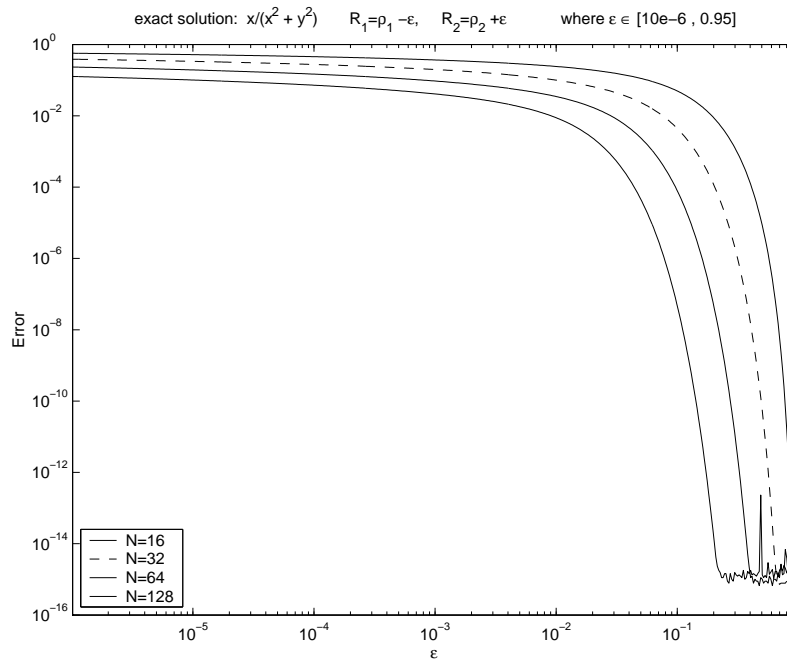


FIGURE 8. Log-log plot of maximum relative error versus ϵ in Example 2

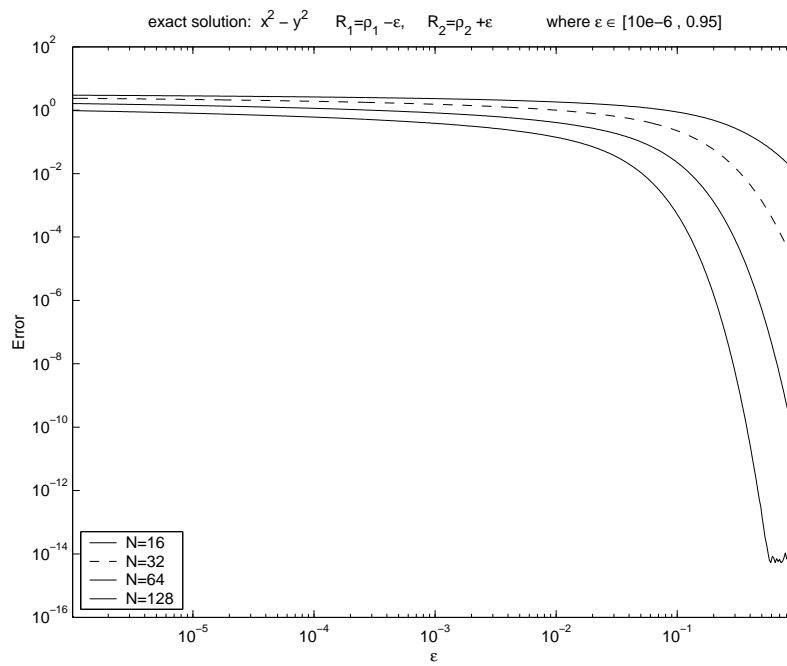
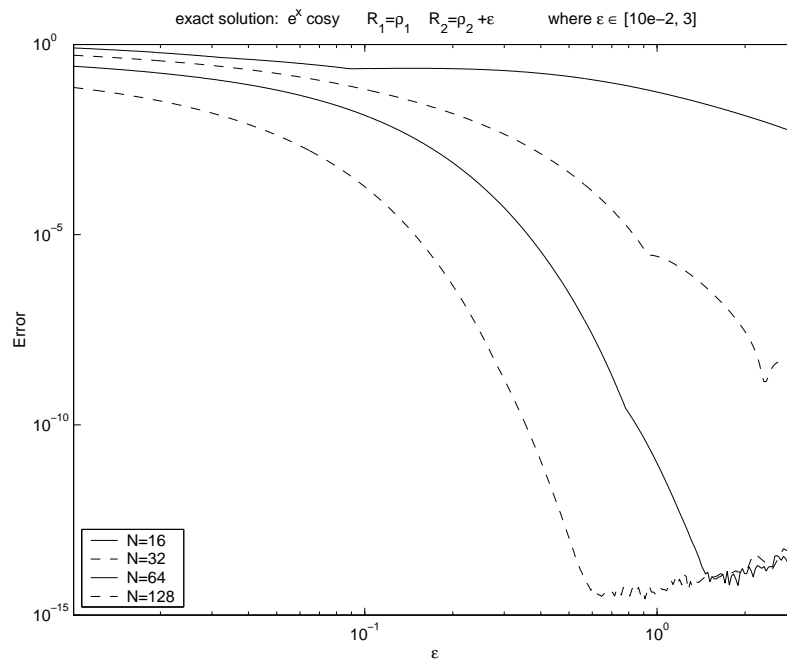


FIGURE 9. Log-log plot of maximum relative error versus ϵ in Example 3

FIGURE 10. Log-log plot of maximum relative error versus ϵ in Example 1