Pergamon

Appl. Math. Lett. Vol. 11, No. 6, pp. 93–99, 1998
© 1998 Elsevier Science Ltd. All rights reserved
Printed in Great Britain
0893-9659/98 $19.00 + 0.00

PII: S0893-9659(98)00109-8

The Effects of Generalized Dispersion on Dissipative Dynamical Systems

Y. S. SMYRLIS
Department of Mathematics and Statistics
University of Cyprus
P.O. Box 537, 1678 Nicosia, Cyprus

D. T. PAPAGEORGIOU
Department of Mathematics
Center for Applied Mathematics and Statistics
New Jersey Institute of Technology
Newark, NJ 07102, U.S.A.

(Received August 1997; accepted September 1997)

Communicated by P. D. Lax

Abstract—We study the effects of dispersion on the Kuramoto-Sivashinsky (KS) equation. In the physical problem considered, there is a full dispersion relation corresponding to a pseudo-differential linear operator added to the KS equation. The long wave limit of this term localizes to a Kortweg-deVries dispersion and we present results from extensive numerical experiments that compare the long time evolution of the global and local systems. It is found that solutions are almost identical in both fixed point (steady traveling waves) and time periodic attractors. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Kuramoto-Sivashinsky, Dispersion, Global attractors.

1. INTRODUCTION

We are concerned with an evolution equation arising in the nonlinear stability of the interface separating two immiscible fluids flowing in a circular pipe, the so-called Core-Annular Flow (CAF). One fluid occupies a core region and the second fluid is in an annular region. Physically, the annular fluid lubricates the motion of the core fluid, thus generating significant technological applications (for example, in lubricated pipelining, see [1]). The evolution equation is derived in the limit of small annulus thickness by the method of matched asymptotic expansions and for a relatively viscous (or, equivalently, slow flowing) core fluid. For details of the derivation and preliminary numerical solutions, see [2]. In scaled form, the evolution equation can be written in the form

$$\begin{gathered} u_t + uu_x + u_{xx} + \nu u_{xxxx} + \frac{\beta}{\nu}\mathcal{L}u = 0, \\ u(x+2\pi,t) = u(x,t), \qquad u(x,0) = u_0(x). \end{gathered} \tag{1}$$

This work was supported by N.A.T.O. Collaborative Research Grant CRG 920097. Additional support for DTP was provided by the National Science Foundation (Grant DMS 9-9-0070), and the work of YSS was supported by a Cyprus University grant. DTP wishes to thank the Department of Mathematics and Statistics, University of Cyprus for its hospitality during his visit.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

We have chosen to scale the equation on 2π-periodic domains, and as a consequence, the length of the system, L say, appears through the parameter $\nu = \pi^2/L^2$. Large systems with extensive complexity are therefore characterized by small ν. Clearly, the limit $\nu \to 0$ is a singular one leading to an ill-posed problem, but it is well known that the KS equation admits increasingly complicated dynamics, including chaos, as ν decreases. For numerical evidence of this, see, for example, [3–6] and references therein. In [4–6], particular routes to chaos are shown to follow the Feigenbaum period doubling scenario. Recent analytical results for the KS, concerning global stability, existence, and estimates on the dimension of attractors can be found in [7–9]. As far as we know, analogous results for the modified equation with dispersion are not available. Nevertheless, numerical experiments suggest that the character of the evolution is not affected by the presence of dispersion but in fact dimensions of limiting attractors decrease.

It is sufficient to study solutions for the nontrivial case of $\nu < 1$. The parameter β is a real constant pertinent to the problem studied in [2], and can be either positive/negative depending on whether the inner fluid is more/less viscous than the outer one, respectively. The operator $\mathcal{L}$ is defined later; it is a pseudo-differential linear operator with an odd purely imaginary spectrum (containing modified Bessel functions) which adds dispersion to KS. Its physical origin in CAFs is in the viscosity difference between the fluids which leads to a viscosity stratification instability. Numerical work in [2] indicates that if the dispersive effects are strong enough (see later for a quantification of this statement), then chaotic long time solutions of KS are organized into traveling wave pulses.

Our main concern in this work is the following question. What is the effect of using a local approximation of $\mathcal{L}$ (i.e., long wave limit yielding the KdV dispersive term—see Section 2) in (1) as opposed to computing with the full dispersion (corresponding to a pseudo-differential operator) as given by the derivation? In conservative systems, such approximations can have significant effects—a classical example arises in nonlinear water waves in the derivation of the Kortweg-deVries (KdV) equation from the Boussinesq equations, where the dispersion relation of the former is found by expanding the full dispersion of the Boussinesq equations for small wavenumber. The KdV equation does not allow for breaking and peaking, whereas model nonlinear systems with full dispersion (such as the dispersion relation of waves on layers of arbitrary depth) can. For further discussion of such systems, see [10, p. 476].

In the context of dissipative-dispersive dynamical systems, models which use KdV dispersion (third derivative) in the KS equation have been studied by various authors; for recent work and a list of numerous references, see [11–13]. These, and earlier studies have exhibited numerically the organizing action of dispersion on unsteady solutions of KS and emergence of traveling wave pulses. The properties and stability of such traveling waves, including detailed bifurcation diagrams, were considered. The routes to chaos for the dissipative/dispersive system have not been mapped out in detail and in particular have not been compared with those of KS. We do not report on such extensive computations here, but instead confine our attention to a detailed comparison of systems with full and long wave dispersion, respectively.

The structure of the article is as follows. In Section 2, we give the evolution equations and derive the long wave dispersion system and identify quantities to be compared. In Section 3, we describe briefly our numerical method and give the results, while Section 4 is devoted to conclusions and future directions.

2. THE KURAMOTO-SIVASHINSKY EQUATION WITH EXACT AND APPROXIMATE DISPERSION

The equation to be solved is (1). The linear dispersion operator $\mathcal{L}$ is given by

$$\mathcal{L}u = i\sum_{-\infty}^{\infty} \mathcal{N}\left(\nu^{1/2}k\right) \hat{u}_k \exp(ikx), \tag{2}$$

where $\hat{u}_k = 1/\pi \int_0^{2\pi} u(x,t)\exp(-ikx)\,dx$ is the usual Fourier transform (in what follows, hats denote Fourier transforms as just defined) of u and the function $\mathcal{N}$ is known from the analysis in [2] and is given by

$$\mathcal{N}(p) = \frac{p^2 I_1^2(p)}{pI_1^2(p) - pI_0^2(p) + 2I_0(p)I_1(p)}, \tag{3}$$

with I_0 and I_1, the modified Bessel functions of order zero and one, respectively. In Fourier space we have, therefore, $\hat{\mathcal{L}u} = i\mathcal{N}(\nu^{1/2}k)\hat{u}_k$ and since $N(-p) = -N(p)$, then $\mathcal{L}$ is a purely dispersive pseudo-differential operator. Given a value of ν, the spectrum of the operator is computed through formula (3) and used in the numerical experiments. The full problem to be solved (termed the Bessel kernel problem) is (1).

Next, we consider a localization of the dispersive operator valid for long waves. This is achieved by a Taylor expansion of (3) for small arguments. The result is

$$\mathcal{N}\left(\nu^{1/2}k\right) = 2\nu^{1/2}k + \frac{1}{6}\nu^{3/2}k^3 + O\left(\nu^{5/2}k^5\right). \tag{4}$$

Substituting (4) into (2) and keeping the first two terms gives

$$\mathcal{L}u \sim \frac{2\beta}{\nu^{1/2}}u_x - \frac{\beta}{6}\nu^{1/2}u_{xxx}. \tag{5}$$

The first term in (5) is removable by a Galilean transformation in a frame moving with speed $2\beta/\nu^{1/2}$ to the right. The evolution equation in this frame is the mixed KS/KdV equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} - \frac{\beta}{6}\nu^{1/2}u_{xxx} = 0 \tag{6}$$

to be solved on periodic domains given initial conditions.

In the numerical work that follows, we fix the value $\beta = -6$ making the coefficient of the dispersive term $\nu^{1/2}u_{xxx}$ in (6) equal to 1. A comparison is made, then, between solutions of (1) and (6) for different values of ν. When ν is larger than approximately 0.04, both equations give traveling waves. We wish to compare the speed, L^2 norm or energy (given by $(\int_0^{2\pi} u^2\,dx)^{1/2}$), and spatial dependence of these profiles. The latter two features are unchanged by the Galilean transformation; however, if the speed of traveling waves of (1) is c_1 and that of (6) is c_2, then the expressions to be compared to ascertain if the two systems give traveling waves with equal speeds, are c_1 and $c_2 - 12/\nu^{1/2}$.

3. NUMERICAL SOLUTIONS AND RESULTS

We surmise the existence of global attractors for the two models of the modified KS equation based on a series of numerical experiments and tests. We have integrated both equations in Fourier space using a pseudo-spectral method. Our code, explained in [5], carries out a split integration of the linear and nonlinear parts of the evolution. The relative precision per time step is 10^{-8}; however, the nature of the observed attractors was unchanged even with relative precision 10^{-5}. The truncation of the Fourier coefficients was based on our preliminary numerical experiments. We have used at least $10\nu^{-1/2}$ modes for each experiment, which means that the neglected modes are below machine precision. Several experiments were carried out to establish existence of windows of ν where the global attractor was a traveling wave (fully modal, bimodal, trimodal, tetramodal, etc.), periodic in time, quasiperiodic, or chaotic with unrecognizable patterns. We have also carried out tests to verify the insensibility of the nature and size of the *approximate attractors* to changes of the time step and number of modes.

In this section, we report on results concerning traveling waves and also on a case with time periodic solutions. The general picture that follows holds for both the Bessel and KdV kernel equations. It is found that there is a globally attracting fully modal traveling wave when ν

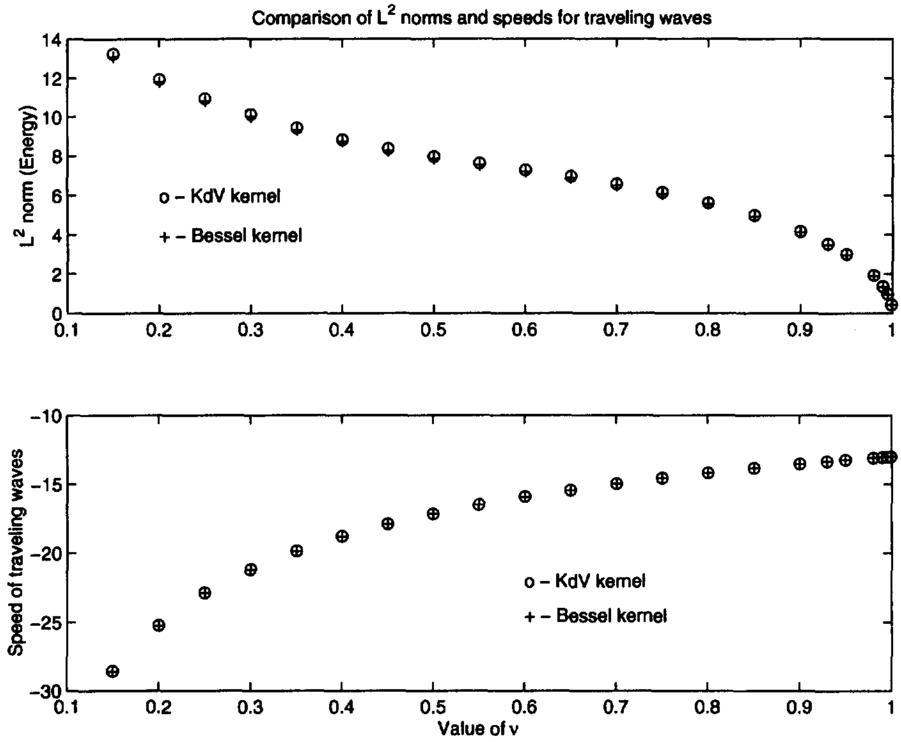

Figure 1. Comparison of L^2 norms and speeds for traveling waves on the primary branch.

Table 1.

ν	Energy$_{\text{Bessel}}$	Energy$_{\text{KdV}}$	Speed$_{\text{Bessel}}$	Speed$_{\text{KdV}}$
0.999	0.429177	0.433972	−12.994322	−13.005004
0.995	0.957997	0.968711	−13.014596	−13.025099
0.99	1.351848	1.366991	−13.040118	−13.050399
0.95	2.969742	3.003401	−13.251796	−13.260320
0.9	4.105875	4.153154	−13.536723	−13.543100
0.8	5.543404	5.609518	−14.186874	−14.189111
0.7	6.483797	6.564181	−14.973045	−14.971322
0.6	7.189423	7.281886	−15.944889	−15.939337
0.5	7.854155	7.956730	−17.180699	−17.171374
0.4	8.727163	8.836685	−18.822286	−18.808885
0.3	10.002748	10.120529	−21.202050	−21.182322
0.2	11.780957	11.916112	−25.212696	−25.181534

decreases just below 1. This primary branch persists to values of ν approximately equal to 0.11 where a bimodal traveling wave attractor coexists. The bimodal traveling waves derive from those on the primary branch by the following similarity transformation. Denoting a traveling wave of either (1) or (6), having speed c at viscosity ν by $U(x; c, \nu)$, then given an integer N, $NU(Nx; Nc, \nu/N^2)$ is also a traveling wave solution (this follows easily from the form of the equations and the form of the kernel). The primary branch, then, implies the existence of multimodal traveling waves supported on ν subwindows of decreasing length. Our computations are able to pick out the stable ones. Such solutions for the KS were given in [5].

In addition, trimodal, tetramodal, and octamodal solutions were also found with much smaller basins of attraction. Convergence to fully modal or bimodal solutions, for example, when they coexist is achieved by suitably varying the initial conditions. At lower values of ν (about 0.04) time periodic solutions are observed, and at lower values quasiperiodic and chaotic solutions are found.

In the results presented next, we make a detailed comparison of solutions of the two equations on the fully modal traveling wave branch. Figure 1 shows a comparison of L^2 norms and speeds of resulting traveling waves, for different values of ν. In comparing speeds, we have corrected the KdV kernel results for the Galilean transformation (see the end of Section 2). Graphically, the two sets of results are almost indistinguishable. Table 1 provides numerical data from Figure 1.

The L^2 norms provide strong evidence that the solutions are close to each other. A pointwise comparison of solutions has also been made for two different values of ν, 0.5, and 0.2. Since long time solutions are traveling waves, stored profiles at the end of a particular computation will generally be out of phase, and must be shifted before comparison. Such comparisons, after shifting, are shown in Figure 2. The top part of the figure corresponds to $\nu = 0.5$ and the lower part to $\nu = 0.2$. Dots outline the KdV kernel profiles and a solid line the Bessel kernel profiles. The profiles are again seen to be almost identical.

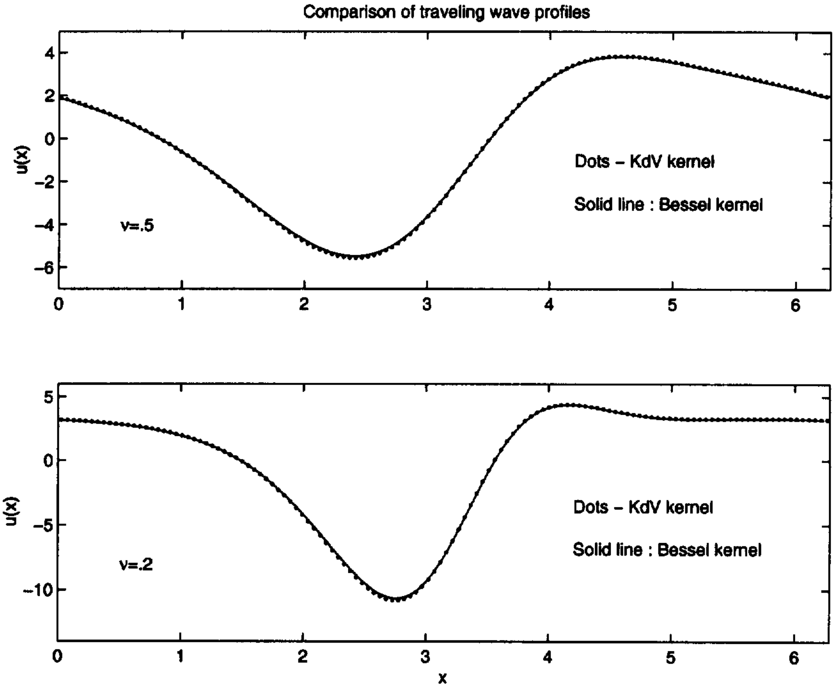

Figure 2. Comparison of traveling wave profiles. Top: $\nu = 0.5$, dots—KdV kernel, solid line—Bessel kernel. Bottom: $\nu = 0.2$, dots—KdV kernel, solid line—Bessek kernel.

Finally, we turn to a case with $\nu = 0.022$ which has time periodic solutions. Figure 3 shows the L^2 norms and their corresponding phase planes. The difference between energy maxima and minima is relatively small as seen from the figure; for the Bessel kernel this range is 35.3298 to 35.3880, while for the KdV kernel it is from 35.7413 to 35.7989. The corresponding periods are estimated to be 27.465 and 28.364, respectively. There appears to be a vertical shift in the mean value of the energy, then, but the features of the solution are similar (i.e., the shape of the time history). This is seen more clearly in the accompanying phase planes which are strikingly similar.

4. CONCLUSIONS

We have established, through a series of numerical experiments, that the KS equation modified by dispersion is quite insensitive to the large wavenumber spectrum of the dispersion relation. The dispersion operator can be approximated by its long wave expansion and this leads to a

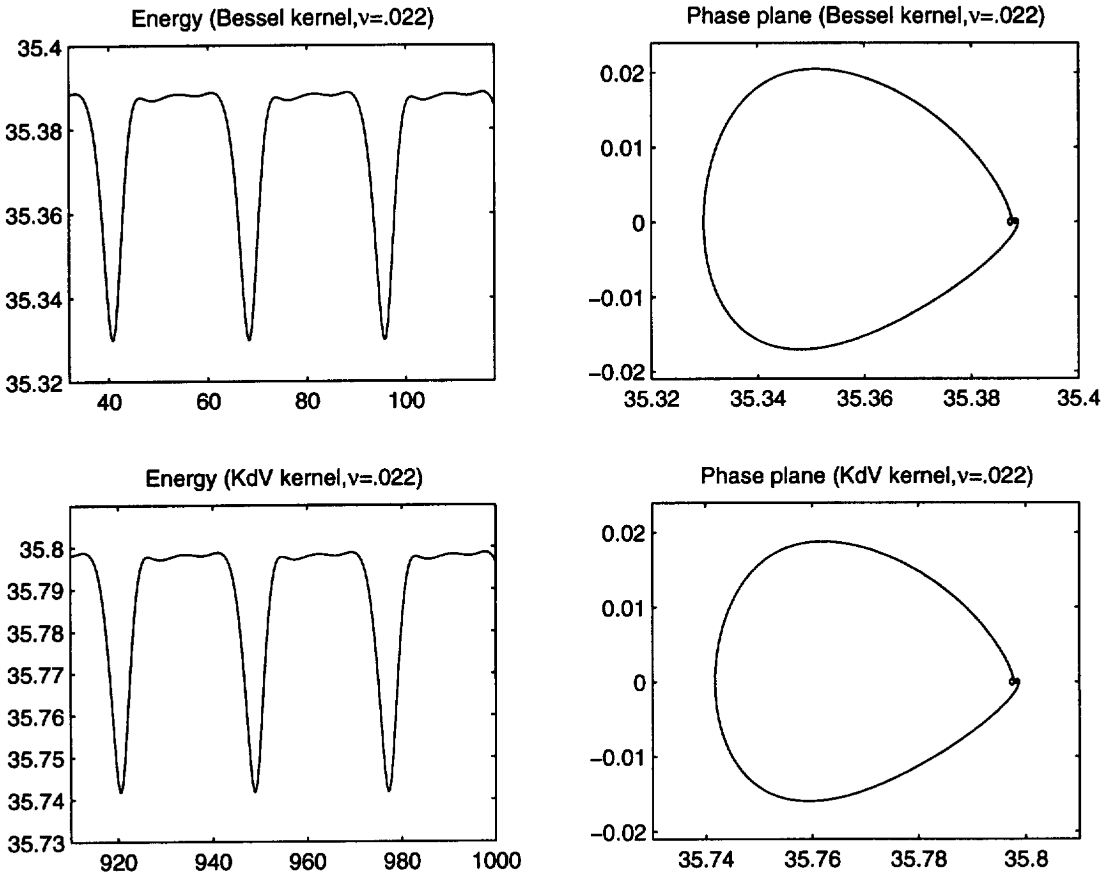

Figure 3. Comparison in a time-periodic attractor, $\nu = 0.022$. L^2 norms and their corresponding phase planes. Top: Bessel kernel. Bottom: KdV kernel.

generic third derivative term augmenting the KS equation. Our comparisons show almost identical solutions in windows which support traveling or time-periodic waves. Direct comparisons become difficult in chaotic attractors due to sensitive dependence on initial conditions, but our experiments indicate that the KS/KdV model equation (6) is quite appropriate for other more complicated systems with generalized dispersion relations. This is probably due to the dissipative nature of the attractors, even though proofs of this are not presently available.

REFERENCES

1. D.D. Joseph and Y.Y. Renardy, Fundamentals of two-fluid dynamics, In *Part II: Lubricated Transport, Drops and Miscible Liquids*, Springer-Verlag, New York, (1993).
2. D.T. Papageorgiou, C. Maldarelli and R.S. Rumschitzki, Nonlinear interfacial stability of core annular film flow, *Phys. Fluids A* **2** (3), 340–352, (1990).
3. J.M. Hyman and B. Nicolaenko, The Kuramoto-Sivashinsky equations, a bridge between PDEs and dynamical systems, *Physica D* **18**, 113–126, (1986).
4. Y.S. Smyrlis and D.T. Papageorgiou, Predicting chaos for infinite dimensional dynamical systems: The Kuramoto-Sivashinsky equation, a case study, *Proc. Nat. Acad. Sci. U.S.A.* **88** (24), 11129–11132, (1991).
5. D.T. Papageorgiou and Y.S. Smyrlis, Computational study of chaotic and ordered solutions of the Kuramoto-Sivashinsky equation, ICASE Report No. 96-12, (1996).
6. D.T. Papageorgiou and Y.S. Smyrlis, Computer assisted study of strange attractors of the Kuramoto-Sivashinsky equation, *Zeitschrift für Angewandte Mathematik ünd Mechanik (ZAMM)* **76** (S2), 57–60, (1996).
7. P. Collet, J.-P. Eckmann, H. Epstein and J. Stubbe, A global attracting set for the Kuramoto Sivashinsky equation, *Commun. Math. Phys.* **152**, 203–214, (1993).
8. J.S. Il'yashenko, Global analysis of the phase portrait for the Kuramoto-Sivashinsky equation, *J. Dyn. Diff. Equations* **4** (4), 585–615, (1992).

9. J. Goodman, Stability of the Kuramoto-Sivashinsky equation and related systems, *Comm. Pure Appl. Math.* **XLVII**, 293–306, (1994).
10. G.B. Whitham, *Linear and Nonlinear Waves*, Wiley-Interscience, New York, (1974).
11. C.I. Christov and M.G. Velarde, Dissipative solitons, *Physica D* **86**, 323–347, (1995).
12. D.E. Bar and A.A. Nepomnyashchy, Stability of periodic waves governed by the modified Kawahara equation, *Physica D* **86**, 586–602, (1995).
13. H.-C. Chang, E.A. Demekhin, D.I. Kopelevich and Y. Ye, Nonlinear wave-number selection in gradient-flow systems, *Phys. Rev. E* **55** (3), 2818–2834, (1997).