Existence and stability of traveling discrete shocks

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Abstract

We are concerned with existence and stability questions for finite difference schemes approximating solutions of scalar conservation laws with shocks. A suitable model for the study of the artifacts created by these schemes near the shocks are the traveling discrete shock profiles; these are discrete shock profiles $\mathbf{v} = (v_k)_{k\in\mathbb{Z}}$ which reappear shifted, when the scheme is applied on them, according to the speed of the shock. Existence of such profiles connecting entropy admissible shocks is already established for monotone schemes, first and third order accurate schemes and the Lax-Wendroff scheme. Jennings showed existence and l^1-stability of these profiles for conservative monotone schemes. Smyrlis showed existence and parametrization by the amount of excess mass and stability for stationary profiles of the Lax-Wendroff scheme. Shih Hsien Yu showed existence of traveling profiles of mild strength for the Lax-Wendroff scheme using inertial manifolds theory. Here we study traveling discrete shock profiles for Lax-Wendroff, Engquist-Osher and monotone schemes. We show existence of such profiles with small shock speed. We also show that these profiles are stable with respect to suitably weighted l^2-norms.

1 Introduction

A scalar conservation law is an initial value problem of the form

$$u_t + f(u)_x = 0 \quad , \quad u(x,0) = u_0(x) \tag{1}$$

where $(x,t) \in \mathbb{R}\times\mathbb{R}_0^+$, u, f scalar, u the conserved quantity and f the flux function, nonlinear in general. The above is one of the simplest quasi-linear partial differential equations and as it is well known, regardless how smooth or small the initial data are, their solutions may very well fail to be smooth for all times. Nevertheless, since conservation laws, derive from integral relations, we could define solutions in a weaker sense, the sense of distributions:

Definition 1.1. *A function $u \in L^\infty(\mathbb{R}\times\mathbb{R}_0^+)$ is called a weak solution of (1) if it satisfies :*

$$\int_{\mathbb{R}} dx \int_{\mathbb{R}_0^+} dt\,(u\phi_t + f(u)\phi_x) + \int_{\mathbb{R}} u_0(x)\phi(x)dx = 0 \tag{2}$$

for all test functions $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}_0^+)$.

Clearly if u is smooth then (1) imply (2). However, if u has a jump discontinuity across the curve $x = \xi(t)$ and

$$\lim_{x\to\xi(t)^+} u(x,t) = u_l \quad , \quad \lim_{x\to\xi(t)^-} u(x,t) = u_r \tag{3}$$

then the definition of the weak solution implies that

$$s = \frac{d\xi}{dt} = \frac{f(u_r) - f(u_l)}{u_r - u_l} \tag{4}$$

and s is the speed of propagation of the discontinuity. Equation (4) is known as the *Rankine-Hugoniot shock condition.*

Formulation (2) does not guarantee uniqueness. More specifically, the same discontinuous initial data may produce weak solutions with shocks, as well as, weak solutions with rarefactions. Nevertheless only one of them has physical relevance. Criteria selecting the right one are called *Entropy conditions.* P.D. Lax in [7] proposed one such criterion which, in the scalar case, reduces to the following formulation :

The shock (u_l, u_r) is admissible whenever

$$f'(u_r) < s < f'(u_l) \tag{5}$$

provided that f is strictly convex.

The simplest admissible discontinuous weak solution, is a *shock*

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x < st \\ u_r & \text{if} \quad x > st \end{cases} \tag{6}$$

with u_l, u_r and s related by (4).

A variety of finite difference schemes have been employed for the approximation of the solution of (1). Such schemes not only provide numerical answers but one anticipates that they could reveal patterns and information about the solutions and the theory to be developed. When a finite difference scheme approximates the solution of a partial differential equation, there is a truncation error expressed by introducing higher order terms in the equation. The form and effects of these artificially introduced terms are related to the order of accuracy of the scheme. In the case of *monotone schemes,*[1] which are first order accurate, the new terms create artificial viscosity and thus smeared shocks. See Figure 1. There is a continuous analogue. The traveling wave solutions of viscous conservation laws converge to a sharp shock as viscosity parameter tends to zero. Extensive study of existence and stability of traveling wave solutions of viscous conservation laws could be found in Il'in & Oleinik [4], S. Kawashima & A. Matsumura [10], Tai-Ping Liu [9] to mention a few.

For second-order accurate schemes the prevalent higher order terms are dispersive. One such example is the Lax-Wendroff scheme [8] :

$$u_k^{n+1} = u_k^n - \frac{\Delta t}{\Delta x}\Big(F(u_k^n, u_{k+1}^n) - F(u_{k-1}^n, u_k^n)\Big) \tag{7}$$

where the numerical flux function is given by

$$F(v,w) = \frac{1}{2}\Big(f(v) + f(w)\Big) - \frac{1}{2}\frac{\Delta t}{\Delta x} f'\left(\frac{v+w}{2}\right)\Big(f(w) - f(v)\Big)$$

and u_k^n approximates the value of $u(n\Delta t, k\Delta x)$. Lax-Wendroff is a nonmonotone scheme in *conservation form*[2]

These numerical artifacts, which persist in approximate solutions of conservation laws near the shocks, were investigated by Gray Jennings [5], who studied discrete shocks. Let $\mathbf{u}^{n+1} = \mathcal{L}_\lambda(\mathbf{u}^n)$ be a finite difference operator where $\mathbf{u}^n = (u_k^n)_{k\in\mathbb{Z}}$, $n \in \mathbb{N}$,

$$\mathcal{L}(\mathbf{u})_k = G_\lambda(u_{k-r}, \ldots, u_{k+r})$$

[1] A finite difference scheme

$$u_k^{n+1} = G(u_{k-r}^n, u_{k-r+1}^n, \ldots, u_{k+r}^n)$$

is called monotone if the partial derivatives $G_{v_\rho}(v_{-r}, v_{-r+1}, \ldots, v_r)$ are positive for all $\rho = -r, \ldots, r$. One such scheme is the Lax-Friedrichs.

[2] A finite difference scheme $u_k^{n+1} = G(u_{k-r}^n, u_{k-r+1}^n, \ldots, u_{k+r}^n)$, $k\in\mathbb{Z}$, $n\in\mathbb{N}$, is said to be in conservation form and thus a *conservative scheme* if

$$G(v_{k-r}^n, v_{k-r+1}^n, \ldots, v_{k+r}^n) =$$
$$v_k^n - \frac{\Delta t}{\Delta x}(F(v_{k-r+1}^n, v_{k-r+1}^n, \ldots, v_{k+r}^n) - F(v_{k-r}^n, v_{k-r+1}^n, \ldots, v_{k+r-1}^n))$$

Remark 1.1. *(i) A conservative scheme is* consistent*, i.e. at least first order accurate, if it satisfies*

$$F(u, u, \ldots, u) = f(u) \qquad \text{[Consistency condition]}$$

(ii) Thus if $(v_k)_{k\in\mathbb{Z}}$ is a profile connecting u_l and u_r then consistency condition and conservation form implies that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}(v_k^{n+1} - v_k^n) &= -\frac{\Delta t}{\Delta x}\sum_{k\in\mathbb{Z}}(F(v_{k-r+1}^n, v_{k-r+1}^n, \ldots, v_{k+r}^n) - F(v_{k-r}^n, v_{k-r+1}^n, \ldots, v_{k+r-1}^n)) \\ &= -\frac{\Delta t}{\Delta x}(f(u_r) - f(u_l)) \ .\end{aligned}$$

Thus the total mass *increases according to the conservation law. in which the artificially introduced dispersive higher order terms create oscillations near the shocks. See Figure 2.*

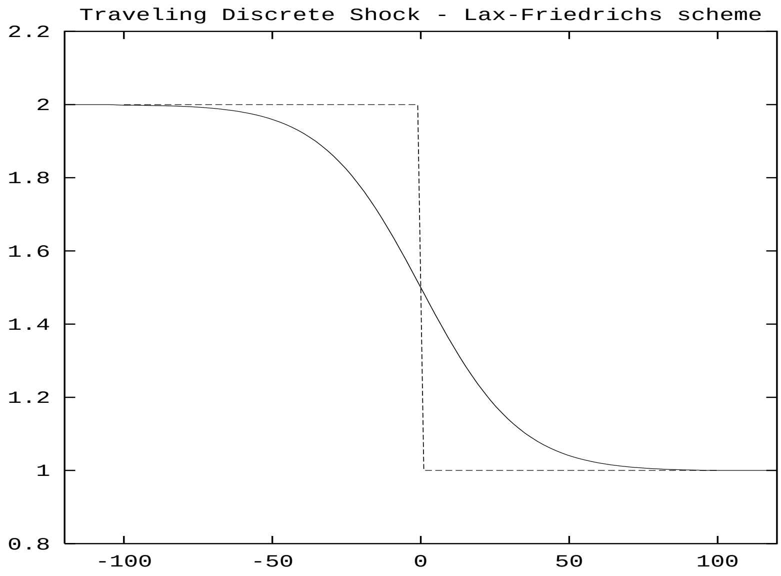

Figure 1: Effects of Numerical viscosity.

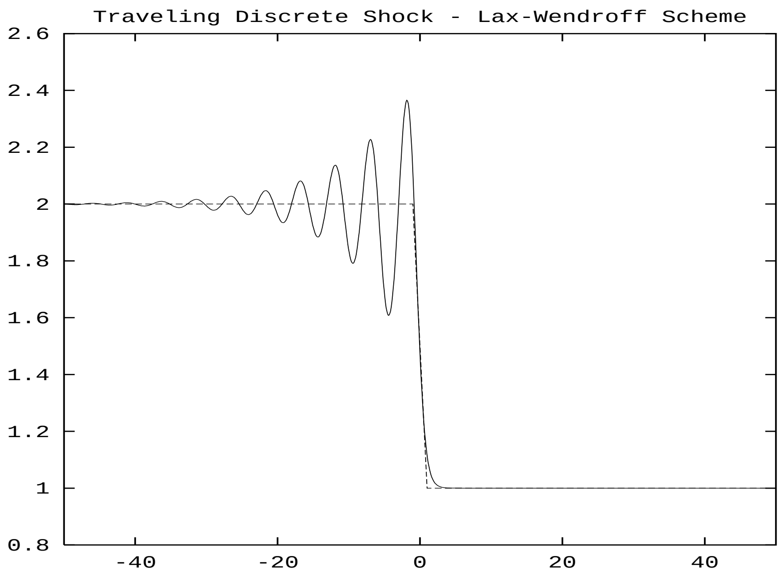

Figure 2: Effects of Numerical dispersion.

and $\lambda = \Delta t/\Delta x$.

Definition 1.2. *A profile* $\mathbf{v} = (v_k)_{k\in\mathbb{Z}}$ *is said to be traveling with speed* s *with respect to the finite difference operator* $\mathcal{L}$ *if* p *iterations of* $\mathcal{L}$ *shift* $\mathbf{v}$ *by* q *positions, i.e.*

$$\mathcal{L}^p(\mathbf{v}) = S_-^q \mathbf{v} \tag{8}$$

where S_- *is the negative shift operator (i.e.* $(S_-\mathbf{v})_k = v_{k-1}$*) and* $q/p = s\Delta t/\Delta x$.

Definition 1.3. *A profile* $\mathbf{v} = (v_k)_{k\in\mathbb{Z}}$ *is said to be a traveling discrete shock profile if it is traveling and it connects the left and right states* u_l *and* u_r *respectively, in the following sense :*

$$\lim_{k\to-\infty} v_k = u_l \qquad \textit{and} \qquad \lim_{k\to+\infty} v_k = u_r \tag{9}$$

Existence and stability of such profiles is significant for the study of the convergence rate of the corresponding scheme to solutions with shocks. In particular they enable us to determine the optimal convergence rate. Detailed discussion appears in Engquist & Yu [2]. Comparison of different such profiles plays an essential role in tracking the exact shock location. Jennings showed, in the case of monotone schemes, that whenever the states u_l and u_r correspond to a shock, there exist traveling discrete shock profiles $\mathbf{v} = \big(v_k\big)_{k\in\mathbb{Z}}$ connecting u_l and u_r, provided that the ratio $\Delta t/\Delta x$ satisfies the *CFL condition.*[3] In [16] we prove the existence of a smooth function $\nu(x)$ such that for every $x\in\mathbb{R}$ the discrete profile

$$\boldsymbol{\nu}^x = \Big(\nu(x+k)\Big)_{k\in\mathbb{Z}} \tag{11}$$

is a traveling discrete shock profile and

$$\sum_{k\le 0}\Big(\nu(x+k) - u_l\Big) + \sum_{k>0}\Big(\nu(x+k) - u_r\Big) = (u_r - u_l)x \tag{12}$$

This function $\nu(x)$ defines a family of traveling discrete shock profiles is thus parametrized by the amount of *excess mass.*

Definition 1.4. *Let* $\mathbf{v} = (v_k)_{k\in\mathbb{Z}}$ *be a discrete profile and* u_l, u_r *be the left and right states respectively. The quantity*

$$\mathcal{M}(\mathbf{v}) = \sum_{k\le 0}(v_k - u_l) + \sum_{k>0}(v_k - u_r) \tag{13}$$

is called excess mass of $\mathbf{v}$ *provided that the above sums converge absolutely.*

[3] CFL or Courant-Friedrichs-Lewy [1], is a necessary condition for numerical stability in hyperbolic problems, requiring that the numerical domain of dependence of the approximate solution contains the domain of dependence of the exact solution. For a three point scheme, i.e. $u_k^{n+1} = G(u_{k-1}^n, u_k^n, u_{k+1}^n)$, approximating a scalar conservation law, the CFL condition reduces to the inequality

$$\frac{\Delta t}{\Delta x}\left|f'\left(u_k^0\right)\right| \le 1 \qquad \text{for all} \quad k\in\mathbb{Z} \tag{10}$$

Clearly if $\mathcal{L}$ is a conservative scheme and $\mathbf{v} = (v_k)_{k\in\mathbb{Z}}$ satisfies (9) then

$$\mathcal{M}(\mathcal{L}(\mathbf{v})) = \mathcal{M}(\mathbf{v}) + \frac{\Delta t}{\Delta x}(f(u_l) - f(u_r)) \tag{14}$$

This structure of $\nu(x)$ allows us to generalize traveling discrete shock profiles:

Definition 1.5. *A continuous function $\nu(x)$, $x\in\mathbb{R}$, is said to be a parametrized traveling discrete shock profile of the finite difference scheme $\mathcal{L}$ connecting the states u_l and u_r, i.e.*

$$\lim_{x\to-\infty} \nu(x) = u_l \qquad , \qquad \lim_{x\to+\infty} \nu(x) = u_r \quad , \tag{15}$$

if it satisfies (12) and $\boldsymbol{\nu}^{x-s\Delta t/\Delta x} = \mathcal{L}(\boldsymbol{\nu}^x)$ for all $x\in\mathbb{R}$ where $\boldsymbol{\nu}^x$ is defined by (11) and u_l, u_r and s satisfy (4).

This parametrization follows a linear relation with the amount of excess mass. Indeed, if $\boldsymbol{\nu}^x$ is the discrete profile defined by (11), then using (13) and (14) we obtain $\mathcal{M}(\boldsymbol{\nu}^x) = (u_r - u_l)x$. Thus, if the traveling discrete shock profiles are *stable*, the limit of the iterations of a perturbed one such profile is not necessarily the same unperturbed one, but instead, a member of the resulting one-parameter family of profiles. Particularly the one with the right amount of excess mass.

Parametrized traveling discrete shock profiles are more convenient since they requires only one application of our operator $\mathcal{L}$ and our problem is reduced to studying the fixed points of $\mathcal{L}$ instead of $\mathcal{L}^p$. Furthermore it allows the speeds of propagation of the discrete shocks to be even *irrational numbers.*

Jennings has also shown l^1-stability of such parametrized profiles for conservative monotone schemes. Michelson [11] and Majda & Ralston [12] showed existence of discrete such profiles for first and third order schemes. Harten, Hyman & Lax [3] observed numerically that such discrete profiles exist and persist even in the case of Lax-Wendroff scheme, which does not belong in the previous categories and its shock profiles are oscillatory. They also realized that these profiles are not l^2-stable. Smyrlis [15] showed existence, parametrization by the amount of excess mass and stability of stationary profiles, i.e. $s = 0$. Stability was obtainable with respect to a weighted l^2-norm. Shih Hsien Yu [17] showed existence of traveling discrete profiles of the Lax-Wendroff scheme in the case of shocks with mild strength. His proof relied on the theory of inertial manifolds.

In [16] we prove existence and stability of traveling discrete shock profiles for Lax-Wendroff and Engquist-Osher [13] schemes.

The stability is with respect to a suitably weighted l^2-norm. Weights are necessary since our linearized finite difference operators are not l^2-stable. These weights depend on the states u_l and u_r. Their introduction deforms the continuous spectrum of the linearized operator in a way that it lies strictly inside the unit circle of the complex plane. We follow the methods of Jones et al [6], Sattinger [14] and Smyrlis [15] in the construction of our weighted norm.

More specifically we prove the following:

Theorem 1.1. *Let $\nu(x)$ be a stable parametrized traveling discrete shock profile of the Lax-Wendroff, Engquist-Osher or of a monotone scheme connecting the states u_l are u_r which correspond to an admissible shock. If $\bar{u}_r$ is sufficiently close to u_r then there exists a parametrized traveling discrete shock profile $\bar{\nu}(x)$ connecting u_l and $\bar{u}_r$. Furthermore this profile is stable with respect to iterations of the finite difference operator in the norm of the weighted space $C_{\alpha,\beta}$ where*

$$C_{\alpha,\beta} = \left\{ u \in C(\mathbb{R}) \;\middle|\; \|u\|_{\alpha,\beta} < +\infty \quad \textit{and} \quad u \textit{ satisfies } (12) \right\}$$

where

$$\|u\|_{\alpha,\beta} < +\infty \max \left\{ \sup_{x<0} \alpha^{-x} |u(x)|, \ \sup_{x \geq 0} \beta^{x} |u(x)| \right\}$$

with the weights $\alpha, \beta > 1$ depending on u_l and u_r .

Remark 1.2. *(i) The existence proof is obtainable using a suitable form of the implicit function theorem. We need that the initial profile $\nu(x)$ is* stable*, which means, stable with respect to the norm of the Banach space $C_{\alpha,\beta}$. Special cases of parametrized traveling discrete shock profiles are obtained in [15] and their stability with respect to norm of $C_{\alpha,\beta}$ is also obtainable. (ii) The weights α, β in the definition of $C_{\alpha,\beta}$ are necessary for stability. Such profiles are known to be unstable with respect to the standard L^2 norm in the case of the Lax-Wendroff scheme. See [3]. The presence of these weights deforms the continuous spectrum of the linearized finite difference operator around the profile in a way that the whole continuous spectrum lies strictly inside the unit circle of the complex plane. See Figure 3.*

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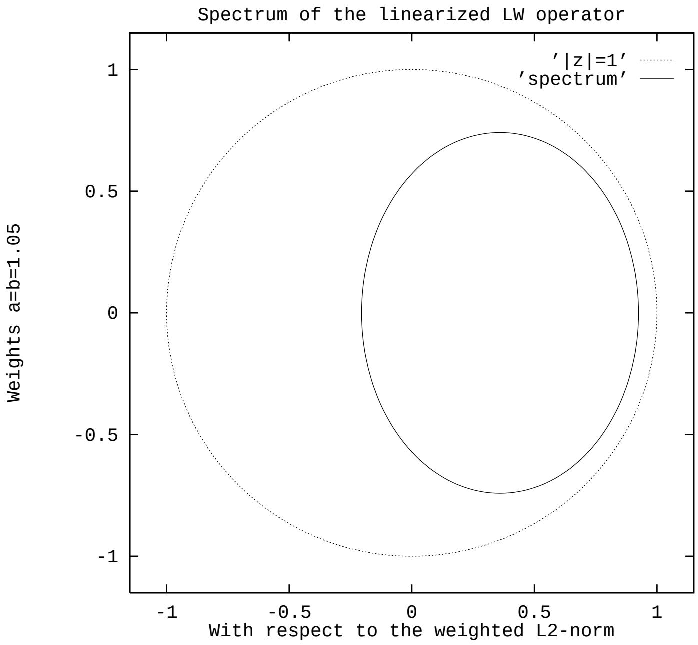

Figure 3: The deformed continuous spectrum of the linearized LW operator.

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