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Robust Approximation of Singularly Perturbed Delay Differential Equations by the *hp* Finite Element Method

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Abstract — We consider the finite element approximation of the solution to a singularly perturbed second order differential equation with a constant delay. The boundary value problem can be cast as a singularly perturbed transmission problem, whose solution may be decomposed into a smooth part, a boundary layer part, an interior/interface layer part and a remainder. Upon discussing the regularity of each component, we show that under the assumption of analytic input data, the *hp* version of the finite element method on an appropriately designed mesh yields *robust* exponential convergence rates. Numerical results illustrating the theory are also included.

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1. Introduction

We consider the approximation of the solution to a singularly perturbed second order differential equation with a constant delay. While *delay differential equations* (DDEs) and their numerical approximation have received much attention (see, e.g., [3] and the references therein), *singularly perturbed* DDEs (SPDDEs) have not. Recently, some articles that start to fill this void have appeared in the literature, such as [2] and [7], in which second order SPDDEs are approximated by finite differences on so-called Shishkin meshes [21]. See also [8] for a Taylor series expansion approach, coupled with finite differences. We also mention the articles [1, 22] which deal with first order SPDDEs and their numerical approximation. It is well known that high order p and hp finite element methods (FEMs) yield excellent results in a variety of settings (see, e.g., [20] and the references therein). Nevertheless, such methods have not been applied to SPDDEs, as of yet, and this is the purpose of this article: to apply an hp FEM to an SPDDE and to prove its robustness and exponential rate of convergence.

The analogous non-delay singularly perturbed problem has been considered by many authors (see, e.g., the books [14, 15, 19] and the references therein). From the point of view of hp FEMs, the monograph [12] gives a complete mathematical treatise along with

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specific guidelines for the construction of the approximation with exponential convergence properties. By extending the known results for non-delay problems to the present case, we are able to prove that the *hp* version of the FEM on carefully designed meshes, yields a robust approximation to the solution of SPDDEs, at an *exponential* rate, independently of the singular perturbation parameter. This is achieved by first writing the SPDDE as a singularly perturbed transmission problem and then using an asymptotic expansion for its solution, that includes a smooth part, a boundary layer part, an interior/interface layer part and a remainder (that is exponentially small). The regularity of each term in the decomposition is studied, in that we present estimates that are explicit in both the singular perturbation parameter, as well as the order of differentiation. This information is then utilized in the approximation scheme to prove that an *hp* FEM on so-called *spectral boundary layer meshes* (cf. [13]) converges exponentially as the degree p of the approximation is increased, when the error is measured in the natural energy norm associated with the boundary value problem.

The rest of the article is organized as follows: In Section 2 we describe the model problem and discuss the typical phenomena. In Section 3 we present the decomposition for the solution and the regularity of each term. Section 4 presents our main approximation result and Section 5 presents the results of some numerical computations verifying our analysis. We end with some conclusions in Section 6.

In what follows, the space of square-integrable functions on an interval $\Omega \subset \mathbb{R}$ will be denoted by $L^2(\Omega)$, with associated inner product

$$(u, v)_\Omega := \int_\Omega uv.$$

We will also utilize the usual Sobolev space notation $H^k(\Omega)$ to denote the space of functions on Ω with $0, 1, 2, \ldots, k$ generalized derivatives in $L^2(\Omega)$, equipped with norm and seminorm $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. We will use the space

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\},$$

where $\partial\Omega$ denotes the boundary of Ω. Finally, the letter C will be used to denote a generic positive constant, independent of any discretization or singular perturbation parameters and possibly having different values in each occurrence.

2. The Model Problem

Consider solving the following problem: Find u^ε such that

$$\begin{cases} -\varepsilon^2 (u^\varepsilon)''(x) + a(x)u^\varepsilon(x) + b(x)u^\varepsilon(x-1) = f(x), \quad x \in \Omega = (0,2), \\ u^\varepsilon(x) = \phi(x), \quad x \in (-1,0), \quad u(2) = L, \end{cases} \tag{1}$$

where a, b, f are given smooth functions on $\overline{\Omega} = [0,2]$, while ϕ is smooth in $[-1,0]$, satisfying

$$a(x) \geqslant \alpha > 0, \quad \beta_0 \leqslant b(x) \leqslant \beta < 0, \quad \alpha + \frac{\beta_0}{2} \geqslant \eta > 0, \quad \forall x \in [0,2], \tag{2}$$

for some constants $\alpha, \beta_0, \beta, \eta$. The parameter $\varepsilon \in (0,1]$ and the constant L in (1) are also given. Delay differential equations arise in a variety of scientific fields, such as biology, ecology, medicine and physics (see, e.g., [4, 6]). Singularly perturbed delay differential equations, like (1), arise for example, in variational problems from control theory [9].

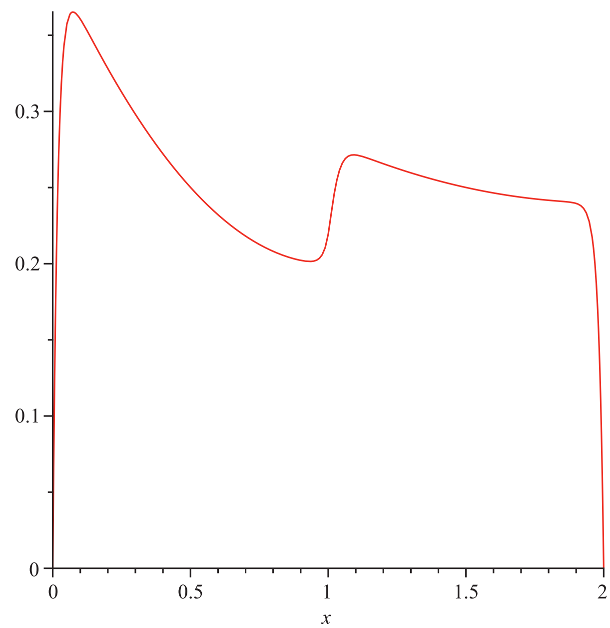

Figure 1. Exact solution $u^\varepsilon(x)$ when $\varepsilon = 0.04$.

If we let $\Omega_- = (0,1), \Omega_+ = (1,2)$ and denote by u_-^ε (respectively u_+^ε) the restriction of u^ε to Ω_- (respectively Ω_+), the above problem is equivalent to the following: Find $(u_-^\varepsilon, u_+^\varepsilon)$ such that

$$\begin{cases} -\varepsilon^2 (u_-^\varepsilon)''(x) + a(x) u_-^\varepsilon(x) = f(x) - b(x)\phi(x-1), \quad x \in \Omega_-, \\ -\varepsilon^2 (u_+^\varepsilon)''(x) + a(x) u_+^\varepsilon(x) + b(x) u_-^\varepsilon(x-1) = f(x), \quad x \in \Omega_+, \\ u_-^\varepsilon(0) = \phi(0), \quad u_+^\varepsilon(2) = L, \\ u_-^\varepsilon(1) = u_+^\varepsilon(1), \quad (u_-^\varepsilon)'(1) = (u_+^\varepsilon)'(1). \end{cases} \tag{3}$$

Without loss of generality we assume that

$$\phi(0) = 0 \quad \text{and} \quad L = 0.$$

The formal limit problem of (3), as $\varepsilon \to 0$, is

$$\begin{cases} a(x) u_-^0(x) = f(x) - b(x)\phi(x-1), \quad x \in \Omega_-, \\ a(x) u_+^0(x) + b(x) u_-^0(x-1) = f(x), \quad x \in \Omega_+, \end{cases}$$

and, in general, the solution u^ε will exhibit boundary/interior layers to the right of $x = 0$, on both sides of $x = 1$ and to the left of $x = 2$ (since there is no reason that $u_\pm^0$ given above satisfy $u_-^0(0) = \phi(0)$, $u_+^0(2) = L$, $u_-^0(1) = u_+^0(1)$ and $(u_-^0)'(1) = (u_+^0)'(1)$). This is illustrated in Figure 1, where the exact solution u^ε is shown, in the case $a(x) = 5$, $b(x) = -1$, $f(x) = 1$, $L = 0$, $\phi(x) = x^2$, $\varepsilon = 0.04$.

Since singularly perturbed transmission problems (and their numerical approximation) have been studied in [16] (see also [10]), we wish to adopt the strategy presented in [16] for this problem. In particular, we expect that the *hp* finite element method on the spectral boundary layer mesh [13]

$$\Delta = \{0, \kappa p\varepsilon, 1 - \kappa p\varepsilon, 1, 1 + \kappa p\varepsilon, 2 - \kappa p\varepsilon, 2\} \tag{4}$$

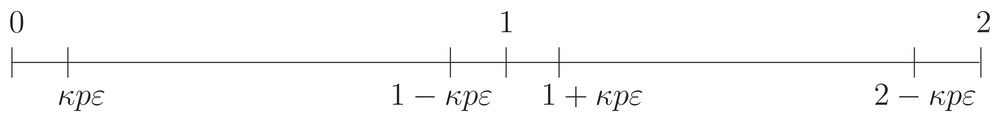

Figure 2. The spectral boundary layer mesh Δ, given by (4).

(see Figure 2) will yield robust exponential convergence – this will be shown in Section 4. (In (4), $\kappa \in \mathbb{R}^+$ and p is the degree of the approximating polynomials.)

In addition to (2), we assume that the functions a, b, f are analytic on $\overline{\Omega}$, ϕ is analytic in $[-1, 0]$, and that there exist constants $C, K_a, K_b, K_f, K_\phi > 0$ such that for any $n = 0, 1, 2, \ldots$

$$\begin{cases} \|a^{(n)}\|_{L^\infty(\Omega)} \leqslant CK_a^n n!, & \|b^{(n)}\|_{L^\infty(\Omega)} \leqslant CK_b^n n!, \\ \|f^{(n)}\|_{L^\infty(\Omega)} \leqslant CK_f^n n!, & \|\phi^{(n)}\|_{L^\infty(-1,0)} \leqslant CK_\phi^n n!. \end{cases} \tag{5}$$

We begin by casting (3) into a variational formulation that reads: Find $u^\varepsilon = (u_-^\varepsilon, u_+^\varepsilon) \in H_0^1(\Omega)$ such that

$$B(u^\varepsilon, v) = F(v) \quad \forall v_\pm \in H_0^1(\Omega_\pm), \tag{6}$$

where

$$\begin{aligned} B(u^\varepsilon, v) = {} & \varepsilon^2 \int_0^1 (u_-^\varepsilon)'(x) v_-'(x) dx + \int_0^1 a(x) u_-^\varepsilon(x) v_-(x) dx \\ & + \varepsilon^2 \int_1^2 (u_+^\varepsilon)'(x) v_+'(x) dx + \int_1^2 a(x) u_+^\varepsilon(x) v_+(x) dx + \int_1^2 b(x) u_-^\varepsilon(x-1) v_+(x) dx, \end{aligned} \tag{7}$$

$$\begin{aligned} F(v) &= \int_0^1 f(x) v_-(x) dx - \int_0^1 b(x)\phi(x-1) v_-(x) dx + \int_1^2 f(x) v_+(x) dx \\ &= \int_0^1 [f(x) - b(x)\phi(x-1)] v_-(x) dx + \int_1^2 f(x) v_+(x) dx. \end{aligned}$$

We associate with the bilinear form (7) the natural *energy norm*, defined as

$$\|v\|_\varepsilon := [B(v,v)]^{1/2} \quad \forall v = (v_-, v_+) \in H_0^1(\Omega). \tag{8}$$

Existence and uniqueness of the solution to (6) follows by the Lax–Milgram lemma. We also have the energy estimate

$$\|u^\varepsilon\|_\varepsilon \leqslant C\{\|f\|_{0,\Omega} + \|\phi\|_{0,\Omega}\}, \tag{9}$$

where the constant $C > 0$ depends *only* on the input data and is *independent* of ε. The approximation u_N^ε to u^ε will come from a finite dimensional subspace V_N of $H_0^1(\Omega)$ such that (6) holds for all $v \in V_N$, with u^ε replaced by u_N^ε. Then, by Céa's lemma

$$\|u^\varepsilon - u_N^\varepsilon\|_\varepsilon \leqslant \inf_{v \in V_N} \|u^\varepsilon - v\|_\varepsilon. \tag{10}$$

The choice of the subspace V_N is discussed in Section 4.

We end this section with the following theorem about the growth of the derivatives of the solution u^ε.

Theorem 2.1. *Let $u^\varepsilon = (u_-^\varepsilon, u_+^\varepsilon)$ be the solution to* (3) *with the data satisfying* (5). *Then there exist constants $C_\pm, K_\pm > 0$ such that for any $n = 0, 1, \ldots$*

$$\|(u_\pm^\varepsilon)^{(n)}\|_{0,\Omega_\pm} \leqslant C_\pm K_\pm^n \max\{\varepsilon^{-1}, n\}^n.$$

Proof. The proof is by induction on n. For $n = 0, 1$, the result follows from (9). Assuming the result holds for n, we establish it for $n+1$ by differentiating the first differential equation in (3) and using the induction hypothesis as well as the assumptions on the data – see the proof of [11, Theorem 1] for details. Once we have the result for u_-^ε, we repeat the procedure for u_+^ε. □

3. Regularity of the Solution via Asymptotic Expansions

Theorem 2.1 gives sufficient information for the approximation of u^ε in the so-called asymptotic range of p, i.e. when $p \geqslant 1/\varepsilon$. For the pre-asymptotic range we will need the decomposition described in this section. Let M be an integer and write

$$u_\pm^\varepsilon = w_M^\pm + u_{\mathrm{BL},M}^\pm + u_{\mathrm{IL},M}^\pm + r_M^\pm, \tag{11}$$

where $w_M^\pm$ denotes the smooth part in $\Omega_\pm$, $u_{\mathrm{BL},M}^\pm$ denotes the boundary layer in $\Omega_\pm$, $u_{\mathrm{IL},M}^\pm$ denotes the interior/transmission layer in $\Omega_\pm$ and $r_M^\pm$ denotes the remainder in the expansion. In the subsections that follow we will analyze each component in (11).

3.1. The Smooth Part

We make the formal ansatz

$$u_\pm^\varepsilon \backsim \sum_{i=0}^{\infty} \varepsilon^i u_i^\pm$$

and insert it into the differential equations in (3), equating like powers of ε. This yields

$$u_0^-(x) = \frac{f(x) - b(x)\phi(x-1)}{a(x)}, \quad x \in (0,1) \tag{12}$$

$$u_0^+(x) = \frac{f(x) - b(x)u_0^-(x-1)}{a(x)}, \quad x \in (1,2), \tag{13}$$

$$u_{2j+1}^\pm(x) = 0, \quad j = 0, 1, 2, \ldots, \tag{14}$$

$$u_{2j+2}^\pm(x) = \frac{(u_{2j}^\pm)''(x)}{a(x)}, \quad j = 0, 1, 2, \ldots . \tag{15}$$

We then define

$$w_M^\pm = \sum_{i=0}^{M} \varepsilon^{2i} u_{2i}^\pm(x), \tag{16}$$

and we note that, with

$$L_\varepsilon u := -\varepsilon^2 u''(x) + a(x)u(x),$$

we have

$$L_\varepsilon w_M^- - (f(x) - b(x)\phi(x-1)) = \varepsilon^{2M+2}(u_{2M}^-)'', \tag{17}$$

$$L_\varepsilon w_M^+ - (f(x) - b(x)u_-^0(x-1)) = \varepsilon^{2M+2}(u_{2M}^+)''. \tag{18}$$

We see that as $\varepsilon \to 0$, $w_M^\pm$ defined by (16) satisfies the differential equations in (3) but neither the boundary conditions at 0 and 2, nor the interface conditions at 0. To correct this, we introduce boundary layer functions in the next subsection.

We have the following result regarding the regularity of $w_M^\pm$.

Theorem 3.1. *There exist constants $C, K_1, K_2 > 0$, independent of ε and depending only on the data, such that under the assumption $0 < 2M\varepsilon K_1 \leqslant 1$,*

$$\|(w_M^\pm)^{(n)}\|_{L^\infty(\Omega_\pm)} \leqslant C K_2^n n! \quad \forall n = 0, 1, 2, \ldots . \tag{19}$$

Proof. See [11, Theorem 3]. □

3.2. The Boundary Layers

Boundary layers $u_{\mathrm{BL},M}^\pm$ are introduced to make up for the fact that the smooth part $w_M^\pm$ does not, in general, satisfy the appropriate boundary conditions (cf. (3)). They are defined via

$$\begin{cases} L_\varepsilon u_{\mathrm{BL},M}^\pm = 0 \text{ in } \Omega_\pm, \\ u_{\mathrm{BL},M}^-(0) = -w_M^-(0), \\ u_{\mathrm{BL},M}^-(1) = u_{\mathrm{BL},M}^+(1), \\ (u_{\mathrm{BL},M}^-)'(1) = (u_{\mathrm{BL},M}^+)'(1), \\ u_{\mathrm{BL},M}^+(2) = -w_M^+(2). \end{cases} \tag{20}$$

The following result gives pointwise bounds on the boundary layer functions and their derivatives.

Theorem 3.2. *Let $u_{\mathrm{BL},M}^\pm$ satisfy (20). Then,*

$$|(u_{\mathrm{BL},M}^-)^{(n)}(x)| \leqslant C K_3^n \max\{n, \varepsilon^{-1}\}^n e^{-\alpha x/\varepsilon} \quad \forall n = 0, 1, 2, \ldots,$$
$$|(u_{\mathrm{BL},M}^+)^{(n)}(x)| \leqslant C K_4^n \max\{n, \varepsilon^{-1}\}^n e^{-\alpha(2-x)/\varepsilon} \quad \forall n = 0, 1, 2, \ldots,$$

with $C, K_3, K_4 > 0$ independent of ε and depending only *on the data.*

Proof. The proof is a direct consequence of [11, Theorem 5]. Indeed, due to the homogeneous transmission conditions

$$u_{\mathrm{BL},M}^-(1) = u_{\mathrm{BL},M}^+(1), \quad (u_{\mathrm{BL},M}^-)'(1) = (u_{\mathrm{BL},M}^+)'(1)$$

at $x = 1$, we can write

$$u_{\mathrm{BL},M}^\pm = -w_M^-(0) u_\varepsilon^- - w_M^+(2) u_\varepsilon^+,$$

where (with the notation of [11]) u_ε^- and u_ε^+ are the smooth solutions of

$$\begin{cases} L_\varepsilon u_\varepsilon^- = 0 \text{ in } \Omega, \\ u_\varepsilon^-(0) = 1, \\ u_\varepsilon^-(2) = 0, \end{cases} \quad \text{and} \quad \begin{cases} L_\varepsilon u_\varepsilon^+ = 0 \text{ in } \Omega, \\ u_\varepsilon^+(0) = 0, \\ u_\varepsilon^+(2) = 1. \end{cases}$$

Since the assumptions on the data guarantee that

$$|w_M^-(0)| + |w_M^+(2)| \leqslant C,$$

for some $C > 0$ independent of ε, we conclude by the estimate (15) (resp. (16)) of [11] on u_ε^- (resp. u_ε^+). □

3.3. The Interior/Interface Layers

Due to (13)–(15) and since the derivatives of u_0^- and $\phi(\cdot - 1)$ do not match at 1, interior/interface layers appear there (see also Figure 1). This phenomenon does not occur in [17] but is encountered in [16] in a slightly different context. Nevertheless as was shown in [16], these interior/interface layers behave just like the boundary layers and are defined via

$$\begin{cases} L_\varepsilon u_{\mathrm{IL},M}^{\pm} = 0 \text{ in } \Omega_\pm, \\ u_{\mathrm{IL},M}^{-}(0) = 0, \\ u_{\mathrm{IL},M}^{+}(1) - u_{\mathrm{IL},M}^{-}(1) = -(w_M^+(1) - w_M^-(1)), \\ (u_{\mathrm{IL},M}^{+})'(1) - (u_{\mathrm{IL},M}^{-})'(1) = -((w_M^+)'(1) - (w_M^-)'(1)), \\ u_{\mathrm{IL},M}^{+}(2) = 0. \end{cases} \tag{21}$$

Analogous to Theorem 3.2, we have the following.

Theorem 3.3. *Let $u_{\mathrm{IL},M}^{\pm}$ satisfy* (21). *Then,*

$$\begin{aligned} |(u_{\mathrm{IL},M}^{-})^{(n)}(x)| &\leqslant C K_5^n \max\{n, \varepsilon^{-1}\}^n e^{-\alpha(1-x)/\varepsilon} \quad \forall n = 0,1,2,\dots, \\ |(u_{\mathrm{IL},M}^{+})^{(n)}(x)| &\leqslant C K_6^n \max\{n, \varepsilon^{-1}\}^n e^{-\alpha(x-1)/\varepsilon} \quad \forall n = 0,1,2,\dots, \end{aligned} \tag{22}$$

with $C, K_5, K_6 > 0$ independent of ε and depending only *on the data.*

Proof. Again the assumptions on the data (see the proof of Theorem 3.2) imply

$$|w_M^+(1) - w_M^-(1)| + |(w_M^+)'(1) - (w_M^-)'(1)| \leqslant C,$$

for some $C > 0$ independent of ε. Accordingly $u_{\mathrm{IL},M}$ is a superposition of v and w via

$$u_{\mathrm{IL},M} = -(w_M^+(1) - w_M^-(1))v - ((w_M^+)'(1) - (w_M^-)'(1))w,$$

where v and w are the respective solutions of

$$\begin{cases} L_\varepsilon v^{\pm} = 0 \text{ in } \Omega_\pm, \\ v^-(0) = 0, \\ v^+(1) - v^-(1) = 1, \\ (v^+)'(1) - (v^-)'(1) = 0, \\ v^+(2) = 0 \end{cases} \tag{23}$$

and

$$\begin{cases} L_\varepsilon w^{\pm} = 0 \text{ in } \Omega_\pm, \\ w^-(0) = 0, \\ w^+(1) - w^-(1) = 0, \\ (w^+)'(1) - (w^-)'(1) = 1, \\ w^+(2) = 0. \end{cases}$$

If we show that $v^+(1)$, $v^-(1)$, $w^+(1)$, $w^-(1)$ are uniformly bounded (with respect to ε), then again the proof finishes by applying [11, Theorem 5].

These boundedness results are proved by using the maximum principle. Indeed denote by H the function

$$H(x) = \begin{cases} 0 & \text{if } x < 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Then we see that $v - H$ is the smooth (i.e. $H^2(0,2)$) solution of

$$\begin{cases} -L_\varepsilon(v - H) = aH \geqslant 0 \text{ in } \Omega, \\ (v - H)(0) = 0, \\ (v - H)(2) = -1. \end{cases}$$

Hence by the maximum principle (see for instance [5, Corollary 3.2]), we get

$$\sup_\Omega (v - H) \leqslant 0,$$

or equivalently

$$v \leqslant H \text{ in } \Omega.$$

In the same manner, taking $\tilde{H} = H - 1$ we see that $v - \tilde{H}$ is the smooth solution of

$$\begin{cases} -L_\varepsilon(v - H) = a\tilde{H} \leqslant 0 \text{ in } \Omega, \\ (v - \tilde{H})(0) = 1, \\ (v - H)(2) = 0. \end{cases}$$

Again by the maximum principle, we get

$$\inf_\Omega (v - \tilde{H}) \geqslant 0,$$

or equivalently

$$v \geqslant \tilde{H} \text{ in } \Omega.$$

We have shown that

$$H - 1 \leqslant v \leqslant H \text{ in } \Omega.$$

Therefore we have proved that

$$-1 \leqslant v^-(1) \leqslant 0 \quad \text{and} \quad 0 \leqslant v^+(1) \leqslant 1.$$

By using the function G defined by

$$G(x) = \begin{cases} 0 & \text{if } x < 1, \\ x - 1 & \text{if } x > 1, \end{cases}$$

we prove, exactly as before, that

$$G - 1 \leqslant w \leqslant G \text{ in } \Omega,$$

and consequently

$$-1 \leqslant w^-(1) \leqslant 0 \quad \text{and} \quad -1 \leqslant w^+(1) \leqslant 0.$$ □

3.4. The Remainder

Finally, we define the remainder via (11) as

$$r_M^{\pm} = u_{\pm}^{\varepsilon} - w_M^{\pm} - u_{\mathrm{BL},M}^{\pm} - u_{\mathrm{IL},M}^{\pm}, \tag{24}$$

and note that

$$\begin{aligned} r_M^{-}(0) &= u_{-}^{\varepsilon}(0) - w_M^{-}(0) - u_{\mathrm{BL},M}^{-}(0) - u_{\mathrm{IL},M}^{-}(0) = 0, \\ r_M^{+}(2) &= u_{+}^{\varepsilon}(2) - w_M^{+}(2) - u_{\mathrm{BL},M}^{+}(2) - u_{\mathrm{IL},M}^{+}(2) = 0. \end{aligned}$$

The next theorem establishes that the energy norm of the remainder is exponentially (in ε) small.

Theorem 3.4. *The remainder, as defined by (24), satisfies*

$$\left(\int_0^2 (\varepsilon^2 |r_M'|^2 + a(x)|r_M|^2)\, dx \right)^{1/2} \leqslant C_1(\varepsilon K(2M+1))^{2M+1}, \tag{25}$$

with $K = \max\{K_b, K_\phi, K_f\}$ which is clearly independent of ε and M; while C_1 is a positive constant (also independent of ε and M) that depends on the constant C appearing in (5) and on α appearing in (2).

Proof. By (3), (17) and (21) we note that

$$\begin{aligned} \|L_\varepsilon r_M^{-}\|_{0,\Omega_-} &= \|L_\varepsilon w_M^{-} - L_\varepsilon u_{-}^{\varepsilon}\|_{0,\Omega_-} \\ &= \varepsilon^{2M+2}\|f - b\phi(\cdot - 1)\|_{0,\Omega_-}. \\ &\leqslant \varepsilon^{2M+2}\big(\|f^{(2M+1)}\|_{0,\Omega_-} + \|(b\phi(\cdot - 1))^{(2M+1)}\|_{0,\Omega_-}\big). \end{aligned}$$

Leibniz's rule yields

$$\|(b\phi(\cdot - 1))^{(2M+1)}\|_{0,\Omega_-} \leqslant \sum_{j=0}^{2M+1} \binom{2M+1}{j} \|b^{(j)}\|_{L^\infty(\Omega_-)} \|\phi^{(2M+1-j)}\|_{L^\infty(-1,0)}.$$

Hence by (5) we get

$$\begin{aligned} \|(b\phi(\cdot - 1))^{(2M+1)}\|_{0,\Omega_-} &\leqslant C^2 \sum_{j=0}^{2M+1} \binom{2M+1}{j} K_b^j K_\phi^{2M+1-j} j!(2M+1-j)! \\ &\leqslant C^2 (K_b + K_\phi)^{2M+1}(2M+1)!, \end{aligned}$$

because $\binom{2M+1}{j} = \frac{(2M+1)!}{j!(2M+1-j)!}$. Using the above estimates and again (5), we get

$$\begin{aligned} \|L_\varepsilon r_M^{-}\|_{0,\Omega_-} &\leqslant C\varepsilon^{2M+2} K^{2M+1}(2M+1)! \\ &\leqslant C(\varepsilon K(2M+1))^{2M+1}, \end{aligned} \tag{26}$$

with $K = \max\{K_b, K_\phi, K_f\}$. Analogously, from (18) we get

$$\begin{aligned} \|L_\varepsilon r_M^{+}\|_{0,\Omega_+} &= \|L_\varepsilon w_M^{+} - L_\varepsilon u_{+}^{\varepsilon}\|_{0,\Omega_+} \\ &\leqslant \varepsilon^{2M+2}\big(\|f^{(2M+1)}\|_{0,\Omega_+} + \|(b u_{-}^{0})^{(2M+1)}\|_{0,\Omega_+}\big). \end{aligned}$$

Since $u_-^0 = \frac{f(x)-b(x)\phi(x-1)}{a(x)}$, we arrive at the same conclusion as above, namely,

$$\|L_\varepsilon r_M^+\|_{0,\Omega_+} \leqslant C_2(\varepsilon K(2M+1))^{2M+1}, \tag{27}$$

with $C_2 > 0$ depending on the constant C appearing in (5). Combining (26) and (27) we get

$$\|L_\varepsilon r_M\|_{0,\Omega} \leqslant \max\{C, C_2\}(\varepsilon K(2M+1))^{2M+1}. \tag{28}$$

Green's formula and the fact that r_M satisfies the homogeneous boundary and interface conditions yield

$$\int_0^2 (\varepsilon^2|r_M'|^2 + a(x)|r_M|^2)\,dx = \int_0^2 (L_\varepsilon r_M)r_M\,dx.$$

Hence by Cauchy–Schwarz's inequality and the assumption $a(x) \geqslant \alpha > 0$ we find

$$\|r_M\|_{0,\Omega} \leqslant \frac{1}{\alpha}\|L_\varepsilon r_M\|_{0,\Omega}.$$

Using this estimate in the previous identity and (28) we get the desired result. □

4. The Approximation of the Solution

We begin this section with the definition of the finite dimensional subspace V_N. Let $\Pi_p(I)$ be the set of polynomials on $I \subset \mathbb{R}$ of degree less than or equal to p and let the spectral boundary layer mesh Δ be given by

$$\Delta = \begin{cases} \{0,1,2\} & \text{when } \kappa p\varepsilon \geqslant 1, \\ \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1, 1+\kappa p\varepsilon, 2-\kappa p\varepsilon, 2\} & \text{when } \kappa p\varepsilon < 1. \end{cases}$$

We then define

$$V_N(\Delta) = \big\{v \in H_0^1(\Omega) : v|_{I_j} \in \Pi_p(I_j)\ \forall j \in \Delta\big\}, \tag{29}$$

where I_j is the jth subinterval in the mesh Δ (cf. (4)). That is, the above space consists of all H_0^1 functions defined on Ω, whose restriction to each subinterval I_j in the mesh Δ is a polynomial of degree p. It follows that

$$N = \dim V_N = \begin{cases} 2p-1 & \text{when } \kappa p\varepsilon \geqslant 1, \\ 6p-1 & \text{when } \kappa p\varepsilon < 1, \end{cases} \tag{30}$$

which is referred to as the number of *degrees of freedom*.

The lemma that follows is proved using Stirling's formula.

Lemma 4.1. *Let* $p \in \mathbb{N}$, $\lambda \in (0,1)$. *Then*

$$\frac{(p-\lambda p)!}{(p+\lambda p)!} \leqslant \Big[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\Big]^p p^{-2\lambda p} e^{2\lambda p+1}.$$

Proof. Using Stirling's approximation

$$\sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n e^{\frac{1}{12n+1}} \leqslant n! \leqslant \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n e^{\frac{1}{12n}} \leqslant \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n e$$

for the factorial (cf. [18]), we have

$$\frac{(p-\lambda p)!}{(p+\lambda p)!} \leqslant \frac{\sqrt{2\pi(1-\lambda)p}}{\sqrt{2\pi(1+\lambda)p}} \frac{(\frac{(1-\lambda)p}{e})^{(1-\lambda)p}}{(\frac{(1+\lambda)p}{e})^{(1+\lambda)p}} \frac{e}{e^{\frac{1}{12(1+\lambda)p+1}}}$$
$$\leqslant \frac{[(1-\lambda)p]^{(1-\lambda)p}}{[(1+\lambda)p]^{(1+\lambda)p}} e^{2\lambda p} e^{1-\frac{1}{12(1+\lambda)p+1}}$$
$$\leqslant \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^p p^{-2\lambda p} e^{2\lambda p} e. \qquad \square$$

We now present our main result.

Theorem 4.2. *Let u^ε be the solution to (3) and $u_N^\varepsilon \in V_N(\Delta)$ its finite element approximation, with V_N given by (29). Then, provided κ is small enough (cf. (34)), there exist constants $C, \sigma > 0$ independent of ε and depending only on the data, such that*

$$\|u^\varepsilon - u_N^\varepsilon\|_\varepsilon \leqslant C N^{3/2} e^{-\sigma N}.$$

Proof. The proof is separated into two cases: when $\kappa p \varepsilon \geqslant 1$ (asymptotic range of p), and when $\kappa p \varepsilon < 1$ (pre-asymptotic range of p).

Case (i): $\kappa p \varepsilon \geqslant \frac{1}{2}, \Delta = \{0, 1, 2\}$.

From [20, Lemma 3.37], we have that there exist interpolants $\mathcal{I}_p u_-^\varepsilon \in \Pi_p(0,1)$, $\mathcal{I}_p u_+^\varepsilon \in \Pi_p(1,2)$ such that

$$\mathcal{I}_p u_-^\varepsilon(0) = u_-^\varepsilon(0), \quad \mathcal{I}_p u_\pm^\varepsilon(1) = u_\pm^\varepsilon(1), \quad \mathcal{I}_p u_+^\varepsilon(2) = \mathcal{I}_p u_+^\varepsilon(2),$$

and

$$\|u_\pm^\varepsilon - \mathcal{I}_p u_\pm^\varepsilon\|_{0,\Omega_\pm}^2 \leqslant \frac{1}{p^2}\frac{(p-s)!}{(p+s)!}\|(u_\pm^\varepsilon)^{(s+1)}\|_{0,\Omega_\pm}^2 \quad \forall s = 0, 1, \ldots, p, \tag{31}$$

$$\|(u_\pm^\varepsilon - \mathcal{I}_p u_\pm^\varepsilon)'\|_{0,\Omega_\pm}^2 \leqslant \frac{(p-s)!}{(p+s)!}\|(u_\pm^\varepsilon)^{(s+1)}\|_{0,\Omega_\pm}^2 \quad \forall s = 0, 1, \ldots, p. \tag{32}$$

Choose $\lambda p - 1 < s \leqslant \lambda p$ for some $\lambda \in (0,1)$ to be selected shortly. Then since $\kappa p \varepsilon \geqslant 1$, we have from Theorem 2.1,

$$\|(u_\pm^\varepsilon)^{(s+1)}\|_{0,\Omega_\pm}^2 \leqslant C_\pm K_\pm^{2(s+1)} \max\{\varepsilon^{-1}, s+1\}^{2(s+1)} \leqslant C_\pm K_\pm^{2(\lambda p+1)} (\lambda p + 1)^{2(\lambda p+1)},$$

provided $\kappa \leqslant \lambda$. Hence, from (31), (32) and the above, we get with the aid of Lemma 4.1,

$$\|u_\pm^\varepsilon - \mathcal{I}_p u_\pm^\varepsilon\|_{0,\Omega_\pm}^2 \leqslant \frac{1}{p^2}\left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^p p^{-2\lambda p} C_\pm (K_\pm e)^{2\lambda p} (\lambda p+1)^2 \Big(\lambda + \frac{1}{p}\Big)^{2\lambda p} p^{2(\lambda p+1)}$$
$$\leqslant p^2 \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^p \Big(K_\pm e\Big(\lambda + \frac{1}{p}\Big)\Big)^{2\lambda p}.$$

So if $p \leqslant 2eK_\pm$, we can take $\lambda = 1/2$ and since

$$\lambda + \frac{1}{p} \leqslant \frac{3}{2}, \quad 2\lambda p \leqslant p \leqslant 2eK_\pm,$$

we have

$$\Big(K_\pm e\Big(\lambda+\frac{1}{p}\Big)\Big)^{2\lambda p} \leqslant \Big(\frac{3K_\pm e}{2}\Big)^{2eK_\pm}.$$

Hence in that case we obtain

$$\|u^\varepsilon_\pm - \mathcal{I}_p u^\varepsilon_\pm\|^2_{0,\Omega_\pm} \leqslant Ce^{-\sigma_\pm p}, \tag{33}$$

with $C, \sigma_\pm > 0$ independent of ε.

On the contrary if $p > 2eK_\pm$, then we choose $\lambda = (2eK_\pm)^{-1} \in (0,1)$, that gives

$$\lambda + \frac{1}{p} \leqslant 2\lambda.$$

Therefore we see that

$$\Big(K_\pm e\Big(\lambda+\frac{1}{p}\Big)\Big)^{2\lambda p} \leqslant (2K_\pm e\lambda)^{2\lambda p} = 1$$

and again we obtain (33).

Note that the constant κ in the definition of the mesh must satisfy

$$\kappa \leqslant (2eK_\pm)^{-1} \tag{34}$$

in the second case. In the first case, this condition obviously implies that $\kappa \leqslant 1/2$.

The same argument may be repeated for (32), yielding

$$\|(u^\varepsilon_\pm - \mathcal{I}_p u^\varepsilon_\pm)'\|^2_{0,\Omega_\pm} \leqslant Cp^2 e^{-\sigma_\pm p}, \tag{35}$$

so that by (8), (10), (33) and (35) we have the desired result.

Case (ii): $\kappa p\varepsilon < \frac{1}{2}$, $\Delta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1, 1+\kappa p\varepsilon, 2-\kappa p\varepsilon, 2\}$.

In this case we make use of the results of Section 3: we begin by decomposing u^ε as in (11) and choose the expansion order M as the integer part of $\kappa p\mu - 1$, where $\mu > 0$ is a fixed constant satisfying $\mu K < 1$, with K the constant in (25). Then, since $\kappa p\varepsilon < 1/2$ and $2M+1 \leqslant 2\kappa p\mu - 1 < 2\kappa p\mu$, we have from (25)

$$\|r_M\|_\varepsilon \leqslant C(\varepsilon K(2M+1))^{2M+1} \leqslant C(\varepsilon K 2\kappa p\mu)^{2\kappa p\mu} \leqslant C(K\mu)^{\mu\kappa p}, \tag{36}$$

which shows, along with stability, that the remainder goes to 0 exponentially fast as p increases.

The remaining terms in the decomposition (11) will be approximated separately. First, for $w^\pm_M$ we have by [20, Lemma 3.37] that there exist $\mathcal{I}_p w^-_M \in \Pi_p(0,1)$, $\mathcal{I}_p w^+_M \in \Pi_p(1,2)$ such that

$$\mathcal{I}_p w^-_M(0) = w^-_M(0), \quad \mathcal{I}_p w^\pm_M(1) = w^\pm_M(1), \quad \mathcal{I}_p w^+_M(2) = \mathcal{I}_p w^+_M(2),$$

and

$$\|w^\pm_M - \mathcal{I}_p w^\pm_M\|^2_{0,\Omega_\pm} \leqslant \frac{1}{p^2}\frac{(p-s)!}{(p+s)!}\|(w^\pm_M)^{(s+1)}\|^2_{0,\Omega_\pm} \quad \forall s = 0,1,\dots,p, \tag{37}$$

$$\|(w^\pm_M - \mathcal{I}_p w^\pm_M)'\|^2_{0,\Omega_\pm} \leqslant \frac{1}{p^2}\frac{(p-s)!}{(p+s)!}\|(w^\pm_M)^{(s+1)}\|^2_{0,\Omega_\pm} \quad \forall s = 0,1,\dots,p. \tag{38}$$

Using (19), we further have from (38),

$$\|(w^\pm_M - \mathcal{I}_p w^\pm_M)'\|^2_{0,\Omega_\pm} \leqslant \frac{1}{p^2}\frac{(p-s)!}{(p+s)!} C K_2^{2(s+1)}[(s+1)!]^2 \quad \forall s = 0,1,\dots,p,$$

so that choosing $p/K_2 - 1 < s \leqslant p/K_2$ and repeating the steps that led to (33) yields

$$\|(w_M^{\pm} - \mathcal{I}_p w_M^{\pm})'\|^2_{0,\Omega_\pm} \leqslant Ce^{-\beta_\pm p}, \tag{39}$$

for some constants $C, \beta_\pm > 0$ independent of ε. The same argument works for (37) and we arrive at

$$\|w_M^{\pm} - \mathcal{I}_p w_M^{\pm}\|^2_{0,\Omega_\pm} \leqslant Ce^{-\beta_\pm p}. \tag{40}$$

We next approximate the boundary and interior/interface layers: since the steps are almost identical for all the terms $u^{\pm}_{\mathrm{BL},M}, u^{\pm}_{\mathrm{IL},M}$, we will only provide details for the approximation of $u^{+}_{\mathrm{IL},M}$, which will be achieved by polynomials of degree p on $[1, 1+\kappa p\varepsilon]$ and by polynomials of degree 1 on $[1+\kappa p\varepsilon, 2]$. From [20, Lemma 3.37], we have that there exists $\mathcal{I}_p u^{+}_{\mathrm{IL},M} \in \Pi_p(1, 1+\kappa p\varepsilon)$, such that

$$\mathcal{I}_p u^{+}_{\mathrm{IL},M}(1) = u^{+}_{\mathrm{IL},M}(1), \quad \mathcal{I}_p u^{+}_{\mathrm{IL},M}(1+\kappa p\varepsilon) = u^{+}_{\mathrm{IL},M}(1+\kappa p\varepsilon),$$

and

$$\|u^{+}_{\mathrm{IL},M} - \mathcal{I}_p u^{+}_{\mathrm{IL},M}\|^2_{0,[1,\kappa p\varepsilon]} \leqslant (\kappa p\varepsilon)^{2s} \frac{1}{p^2} \frac{(p-s)!}{(p+s)!} \|(u^{+}_{\mathrm{IL},M})^{(s+1)}\|^2_{0,[1,\kappa p\varepsilon]} \quad \forall s = 0, 1, \ldots, p, \tag{41}$$

$$\|(u^{+}_{\mathrm{IL},M} - \mathcal{I}_p u^{+}_{\mathrm{IL},M})'\|^2_{0,[1,\kappa p\varepsilon]} \leqslant (\kappa p\varepsilon)^{2s} \frac{(p-s)!}{(p+s)!} \|(u^{+}_{\mathrm{IL},M})^{(s+1)}\|^2_{0,[1,\kappa p\varepsilon]} \quad \forall s = 0, 1, \ldots, p. \tag{42}$$

From (22) we see that

$$\|(u^{+}_{\mathrm{IL},M})^{(s+1)}\|^2_{0,[1,\kappa p\varepsilon]} \leqslant \kappa p\varepsilon C K_4^{2(s+1)} \max\{s+1, \varepsilon^{-1}\}^{2(s+1)} \leqslant \kappa p\varepsilon C K_4^{2(s+1)} \varepsilon^{-(2s+1)},$$

so (41) becomes

$$\begin{aligned}\|u^{+}_{\mathrm{IL},M} - \mathcal{I}_p u^{+}_{\mathrm{IL},M}\|^2_{0,[1,\kappa p\varepsilon]} &\leqslant (\kappa p\varepsilon)^{2s} \frac{1}{p^2} \frac{(p-s)!}{(p+s)!} \kappa p\varepsilon C K_4^{2(s+1)} \varepsilon^{-(2s+1)} \\ &\leqslant C K_4^{2(s+1)} (\kappa p)^{2s+1} \frac{1}{p^2} \frac{(p-s)!}{(p+s)!}.\end{aligned}$$

Choosing $\overline{\lambda} p - 1 < s \leqslant \overline{\lambda} p$ for some $\overline{\lambda} \in (0,1)$ to be selected shortly and using Lemma 4.1, we further get

$$\begin{aligned}\|u^{+}_{\mathrm{IL},M} - \mathcal{I}_p u^{+}_{\mathrm{IL},M}\|^2_{0,[1,\kappa p\varepsilon]} &\leqslant C K_4^{2(\overline{\lambda} p+1)} (\kappa p)^{2\overline{\lambda} p+1} \left[\frac{(1-\overline{\lambda})^{(1-\overline{\lambda})}}{(1+\overline{\lambda})^{(1+\overline{\lambda})}}\right]^p p^{-2\overline{\lambda} p} e^{2\overline{\lambda} p} \\ &\leqslant C K_4^2 e\kappa p \left[\frac{(1-\overline{\lambda})^{(1-\overline{\lambda})}}{(1+\overline{\lambda})^{(1+\overline{\lambda})}}\right]^p (e\kappa K_4)^{2\overline{\lambda} p} \\ &\leqslant Cpe^{-\gamma p},\end{aligned} \tag{43}$$

where (34) was used and $\gamma = |\ln q|$, $q = \frac{(1-\overline{\lambda})^{(1-\overline{\lambda})}}{(1+\overline{\lambda})^{(1+\overline{\lambda})}} < 1$. Repeating the above argument for (42) we arrive at

$$\|(u^{+}_{\mathrm{IL},M} - \mathcal{I}_p u^{+}_{\mathrm{IL},M})'\|^2_{0,[1,\kappa p\varepsilon]} \leqslant Cp^3 \varepsilon^{-1} e^{-\gamma p}. \tag{44}$$

Now, on $[1+\kappa p\varepsilon, 2]$ the function $u^+_{\mathrm{IL},M}$ is already exponentially small, hence we will approximate it by its linear interpolant $\mathcal{I}_1 u^+_{\mathrm{IL},M} \in \Pi_1(1+\kappa p\varepsilon, 2)$. We have

$$\|(u^+_{\mathrm{IL},M})'\|^2_{0,[1+\kappa p\varepsilon,2]} = \int_{1+\kappa p\varepsilon}^{2} [(u^+_{\mathrm{IL},M})']^2 dx \leqslant C\varepsilon^{-2}(1-\kappa p\varepsilon) \max_{x\in[1+\kappa p\varepsilon,2]} \{e^{-2\alpha(x-1)/\varepsilon}\}$$
$$\leqslant C\varepsilon^{-2} e^{-2\alpha(\kappa p\varepsilon)/\varepsilon} \leqslant C\varepsilon^{-2}e^{-2\alpha\kappa p},$$

so that

$$\begin{aligned}\|(u^+_{\mathrm{IL},M} - \mathcal{I}_1 u^+_{\mathrm{IL},M})'\|^2_{0,[1+\kappa p\varepsilon,2]} &\leqslant \|(u^+_{\mathrm{IL},M})'\|^2_{0,[1+\kappa p\varepsilon,2]} + \|(\mathcal{I}_1 u^+_{\mathrm{IL},M})'\|^2_{0,[1+\kappa p\varepsilon,2]} \\ &\leqslant C\varepsilon^{-2}e^{-2\alpha\kappa p}.\end{aligned} \tag{45}$$

Similarly, we get

$$\|u^+_{\mathrm{IL},M} - \mathcal{I}_1 u^+_{\mathrm{IL},M}\|^2_{0,[1+\kappa p\varepsilon,2]} \leqslant Ce^{-2\alpha\kappa p}, \tag{46}$$

and by combining (43), (44), (45) and (46) we obtain

$$\|u^+_{\mathrm{IL},M} - \mathcal{I}u^+_{\mathrm{IL},M}\|^2_{0,\Omega_+} \leqslant Ce^{-\sigma p}, \quad \|(u^+_{\mathrm{IL},M} - \mathcal{I}u^+_{\mathrm{IL},M})'\|^2_{0,\Omega_+} \leqslant C\varepsilon^{-2}e^{-\sigma p}, \tag{47}$$

for some constants $C, \sigma > 0$ independent of ε and with

$$\mathcal{I} = \begin{cases} \mathcal{I}_p & \text{on } [1, 1+\kappa p\varepsilon], \\ \mathcal{I}_1 & \text{on } [1+\kappa p\varepsilon, 2]. \end{cases}$$

As mentioned earlier, the approximation of the terms $u^-_{\mathrm{IL},M}, u^\pm_{\mathrm{BL},M}$ is identical to the above, hence

$$\|u^-_{\mathrm{IL},M} - \mathcal{I}u^-_{\mathrm{IL},M}\|^2_{0,\Omega_+} \leqslant Ce^{-\sigma p}, \qquad \|(u^-_{\mathrm{IL},M} - \mathcal{I}u^-_{\mathrm{IL},M})'\|^2_{0,\Omega_+} \leqslant C\varepsilon^{-2}e^{-\sigma p}, \tag{48}$$
$$\|u^\pm_{\mathrm{BL},M} - \mathcal{I}u^\pm_{\mathrm{BL},M}\|^2_{0,\Omega_\pm} \leqslant Ce^{-\sigma p}, \qquad \|(u^\pm_{\mathrm{BL},M} - \mathcal{I}u^\pm_{\mathrm{IL},M})'\|^2_{0,\Omega_+} \leqslant C\varepsilon^{-2}e^{-\sigma p}. \tag{49}$$

Therefore, with $\mathcal{I}^\varepsilon$ defined as

$$\mathcal{I}^\varepsilon u^\varepsilon := \mathcal{I}_p w^\pm_M + \mathcal{I}u^\pm_{\mathrm{BL},M} + \mathcal{I}u^\pm_{\mathrm{IL},M},$$

we have from (11),

$$\begin{aligned}\|u^\varepsilon - \mathcal{I}^\varepsilon u^\varepsilon\|^2_{0,\Omega_\pm} &= \|(w^\pm_M + u^\pm_{\mathrm{BL},M} + u^\pm_{\mathrm{IL},M} + r^\pm_M) - (\mathcal{I}_p w^\pm_M + \mathcal{I}u^\pm_{\mathrm{BL},M} + \mathcal{I}u^\pm_{\mathrm{IL},M})\|^2_{0,\Omega_\pm} \\ &\leqslant \|w^\pm_M - \mathcal{I}_p w^\pm_M\|^2_{0,\Omega_\pm} + \|u^\pm_{\mathrm{BL},M} - \mathcal{I}u^\pm_{\mathrm{BL},M}\|^2_{0,\Omega_\pm} \\ &\quad + \|u^\pm_{\mathrm{IL},M} - \mathcal{I}u^\pm_{\mathrm{IL},M}\|^2_{0,\Omega_\pm} + \|r^\pm_M\|^2_{0,\Omega_\pm} \\ &\leqslant Cpe^{-\sigma p},\end{aligned} \tag{50}$$

where (36), (40), (47), (48) and (49) were used. Similarly, using (36), (39), (47), (48) and (49), we get

$$\begin{aligned}\varepsilon^2\|(u^\varepsilon - \mathcal{I}^\varepsilon u^\varepsilon)'\|^2_{0,\Omega_\pm} &\leqslant \|(w^\pm_M, -\mathcal{I}_p w^\pm_M)'\|^2_{0,\Omega_\pm} + \|(u^\pm_{\mathrm{BL},M} - \mathcal{I}u^\pm_{\mathrm{BL},M})'\|^2_{0,\Omega_\pm} \\ &\quad + \|(u^\pm_{\mathrm{IL},M} - \mathcal{I}u^\pm_{\mathrm{IL},M})'\|^2_{0,\Omega_\pm} + \|(r^\pm_M)'\|^2_{0,\Omega_\pm} \\ &\leqslant Cp^3e^{-\sigma p}.\end{aligned} \tag{51}$$

From (8), (10), (50) and (51) we get the desired result. □

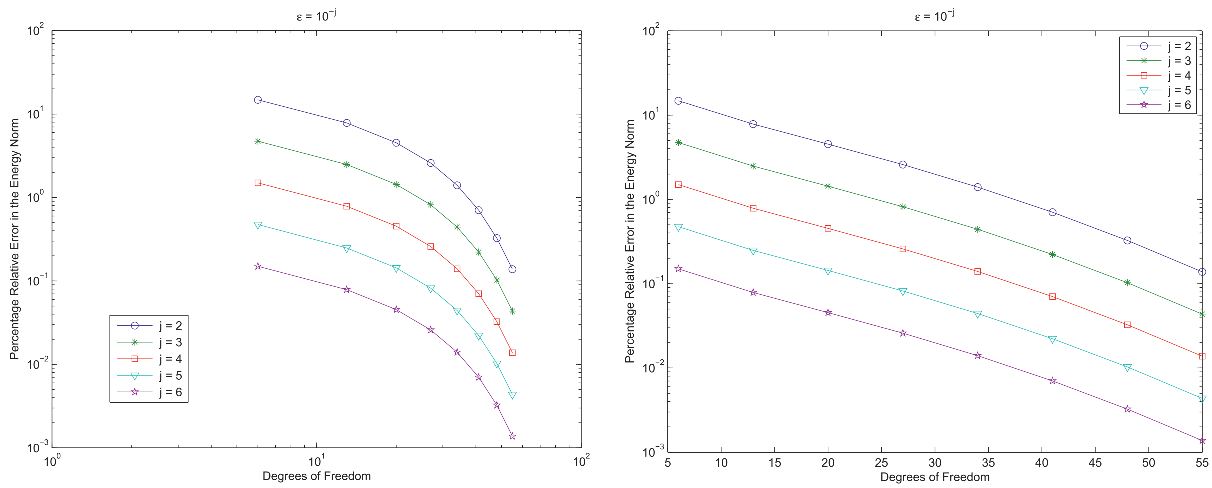

Figure 3. Energy norm convergence for the *hp* version. Left: Log-log scale. Right: Semi-log scale.

5. Numerical Results

We now present the results of numerical computations verifying our theoretical findings, as well as comparing the proposed method to other commonly used ones found in the literature. We consider problem (1) with $a(x) = 5$, $b(x) = -1$, $f(x) = 1$, $L = 0$, $\phi(x) = 0$, and exact solution known; knowledge of the exact solution makes our computations reliable. We will be interested in the percentage relative error in the energy norm

$$E := 100 \times \frac{\|u^\varepsilon - u^\varepsilon_{FEM}\|_\varepsilon}{\|u^\varepsilon\|_\varepsilon},$$

and we will be plotting it versus the number of degrees of freedom N (cf. (30)) in log-log and semi-log scales.

We use the proposed method on the spectral boundary layer mesh Δ given by (4), with $\kappa = 1$ and $p = 1, \ldots, 8$. Figure 3 shows the performance of the method, for several values of ε; the robustness and exponential convergence are readily visible. In fact, as $\varepsilon \to 0$, the method not only does not deteriorate (i.e. it is robust), it actually performs better – see, e.g., [20] for an explanation of this phenomenon for non-delay singularly perturbed problems.

In Figure 4 we show the performance of various methods for $\varepsilon = 10^{-6}$: the h version with piecewise linears on a uniform mesh (known not to converge uniformly), the h version with polynomials of degree 1 and 2, on a (piecewise uniform) Shishkin mesh [21], and the proposed hp method on the spectral boundary layer mesh. As expected, the first method does not perform well, since the mesh does not incorporate ε in its definition. The h version on the Shishkin mesh is performing uniformly, at the quasi-optimal rate $O((N^{-1} \ln N)^p)$. This figure clearly shows how competitive the hp FEM is, even for this class of problems.

6. Conclusions

We have proved the robustness and exponential convergence rate of an hp finite element method on spectral boundary layer meshes, applied to a second order SPDDE. The problem was first written as a singularly perturbed transmission problem, whose solution admits an asymptotic expansion that includes a smooth part, a boundary layer part, an interior/interface layer and a remainder. Bounds on the derivatives of each part allowed us

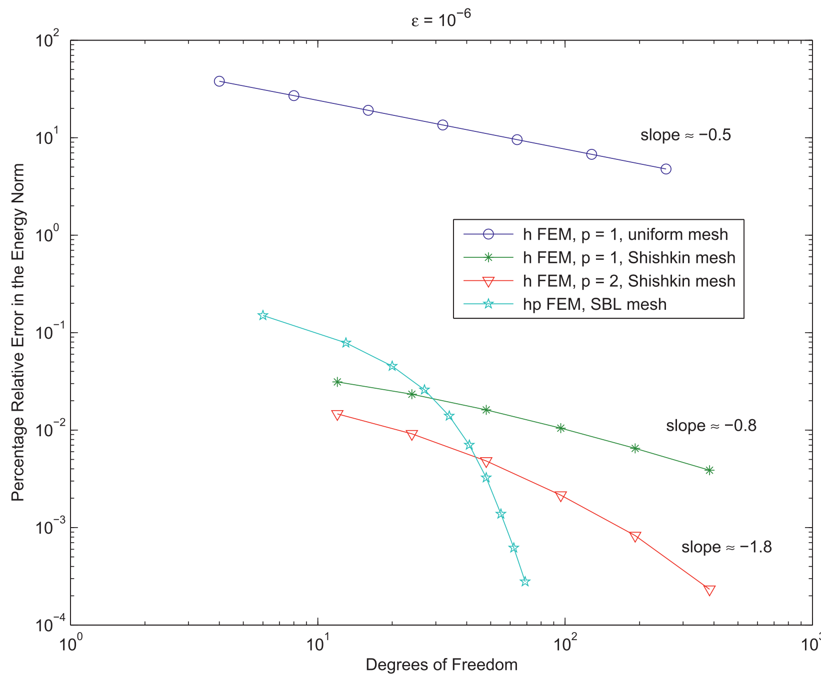

Figure 4. Comparison of various methods, for $\varepsilon = 10^{-6}$.

to prove the desired result. The current work extends our previously published results [16], to the case of SPDDEs, which involve interface layers due to the fact that u_0^- and $\phi(\cdot - 1)$ (cf. (12)) do not match at the interface point $x = 1$. Numerical computations agree with our theoretical findings and provide evidence that place the proposed method among the state-of-the-art for SPDDEs.

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